

G-GROUPS OF COHEN-MACAULAY RINGS WITH n -CLUSTER TILTING OBJECTS

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ABSTRACT. Let (R, \mathfrak{m}, k) denote a local Cohen-Macaulay ring such that the category of maximal Cohen-Macaulay R -modules $\mathbf{mcm} R$ contains an n -cluster tilting object L . In this paper, we compute $G_1(R) := K_1(\mathbf{mod} R)$ explicitly as a direct sum of a free group and a specified quotient of $\mathrm{Aut}_R(L)_{\mathrm{ab}}$ when R is a k -algebra and k is algebraically closed. Moreover, we give some explicit computations of $\mathrm{Aut}_R(L)_{\mathrm{ab}}$ and $G_1(R)$ for certain hypersurface singularities.

1. INTRODUCTION

Throughout this section (R, \mathfrak{m}, k) will always denote a local Cohen-Macaulay ring. Since the introduction of higher algebraic K-theory by Quillen there has been a significant effort to understand the structure of the K -groups $K_i(\mathcal{A})$, for \mathcal{A} an exact category. Our particular interest is when $\mathcal{A} = \mathbf{mod} R$, the category of finitely generated R -modules. The groups $K_i(\mathbf{mod} R)$ are denoted by $G_i(R)$. They are, unsurprisingly, called the G -groups of R (they are also called K' -groups in the literature and may be denoted by $K'_i(R)$). In Section 2, we will discuss notation and various definitions of K -groups needed in the computation of $G_1(R)$.

Let $\mathbf{proj} R$ be the subcategory of $\mathbf{mod} R$ of finitely generated projective R -modules. Now the inclusion $\mathbf{proj} R \hookrightarrow \mathbf{mod} R$ induces a map of groups between $K_i(R) := K_i(\mathbf{proj} R)$ and $G_i(R)$. It is of interest to understand the properties of this induced homomorphism. In particular, when is this map an isomorphism? This is precisely the case when R is regular, following immediately from Quillen's Resolution Theorem ([15], §Theorem 3). However, regular local rings are exceptionally well-behaved, so one cannot expect this behavior in general. Suppose $i = 0$. It is well-known $K_0(R)$ is isomorphic to \mathbb{Z} (see ([16], Theorem 1.3.11)), but what of $G_0(R)$? It is well-known that $G_0(R)$ is just the Grothendieck group of the category $\mathbf{mod} R$. Moreover, if we impose the condition that R has *finite Cohen-Macaulay type* (that is, there are, up to isomorphism, finitely many indecomposable maximal Cohen-Macaulay R -modules) then the structure of $G_0(R)$ is elucidated in its entirety by the following theorem.

Theorem 1.1. ([21], Theorem 13.7)

Suppose there are t non-free indecomposable maximal Cohen-Macaulay R -modules and denote by \mathfrak{S} the free abelian group on the set of isomorphism classes of indecomposable maximal Cohen-Macaulay R -modules. The map $\mathfrak{S} \rightarrow G_0(R)$ given by $X \mapsto [X]$ is surjective and its kernel is generated by

$$\{X - X' - X'' \mid \exists \text{ an Auslander-Reiten sequence } 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0\}$$

And $G_0(R) \cong \mathrm{coker}(\Upsilon)$, where $\Upsilon : \mathbb{Z}^{\oplus t} \rightarrow \mathbb{Z}^{\oplus(t+1)}$ is the Auslander-Reiten homomorphism.

The immense usefulness of Theorem 1.1 lies in the fact that the computation of $G_0(R)$ has been reduced to linear algebra, as the Auslander-Reiten homomorphism can be readily computed from the Auslander-Reiten quiver. This quickly leads to the explicit computation of $G_0(R)$ for all simple singularities of finite type (see [21], Proposition 13.10). One can quickly see that these groups are often not \mathbb{Z} .

Moving up one rung on the K -theory ladder, it is well-known that $K_1(R) := K_1(\mathbf{proj} R) \cong R^*$ (see ([18], Example 1.6)). However, the structure of $G_1(R)$ was not known for some time until the work of H. Holm in [7] and V. Navkal in [13]. In the former, computing $G_1(R)$ was carried out over an R which has finite Cohen-Macaulay type and it was found that $G_1(R)$ could be computed as

an explicit quotient of $\text{Aut}_R(L)_{\text{ab}}$, with L an additive generator for the category maximal Cohen-Macaulay R -modules, $\mathbf{mcm} R$ (noting such an L exists if and only if R has finite Cohen-Macaulay type). The latter produced the following theorem.

Theorem 1.2. ([13], Theorem 1.1)

Assume that R is Henselian and the category $\mathbf{mcm} R$ has an n -cluster tilting object L . Let \mathfrak{J} be the set of isomorphism classes of indecomposable summands of L and set $\mathfrak{J}_0 = \mathfrak{J} \setminus \{R\}$. Then there is a long exact sequence

$$\cdots \longrightarrow \bigoplus_{M \in \mathfrak{J}_0} G_i(\kappa_M) \longrightarrow G_i(\Lambda) \longrightarrow G_i(R) \longrightarrow \bigoplus_{M \in \mathfrak{J}_0} G_{i-1}(\kappa_M) \longrightarrow \cdots$$

Where

$$\Lambda = \text{End}_R(L)^{op} \quad \text{and} \quad \kappa_M = \text{End}_R(M)^{op} / \text{rad}(\text{End}_R(M))^{op}$$

κ_M is always a division ring, and when $R/\mathfrak{m} = k$ is algebraically closed, $\kappa_M = k$.

The long exact sequence ends in presentation

$$\bigoplus_{M \in \mathfrak{J}_0} G_0(\kappa_M) \longrightarrow G_0(\Lambda) \longrightarrow G_0(R) \longrightarrow 0$$

of $G_0(R)$.

While the definition of an n -cluster tilting object for $\mathbf{mcm} R$ is technical (see Definition 2.16), we remark that $\mathbf{mcm} R$ has 1-cluster tilting object if and only if it has finite Cohen-Macaulay type. Since $G_0(\Lambda) = \mathbb{Z}^{\mathfrak{J}}$ (See Lemma 3.2) and $\bigoplus_{M \in \mathfrak{J}_0} G_0(\kappa_M) = \mathbb{Z}^{\mathfrak{J}_0}$, the presentation of $G_0(R)$ given above

is precisely the one given in Theorem 1.1 when L is an additive generator of $\mathbf{mcm} R$. That is, Theorem 1.2 generalizes Theorem 1.1. In fact, we show in Section 3 that utilizing Theorem 1.2 and techniques from [7], we can generalize and simplify the results [7] on the structure of $G_1(R)$. Our main theorem is the following, with notation as in Theorem 1.2.

Theorem 1.3. Suppose that R is a k -algebra, $\text{char}(k) \neq 2$ and k is algebraically closed. If $\mathbf{mcm} R$ has an n -cluster tilting object L such that Λ has finite global dimension, we can compute explicitly a subgroup Ξ of $\text{Aut}_R(L)_{\text{ab}}$ such that

$$G_1(R) \cong H \oplus \text{Aut}_R(L)_{\text{ab}} / \Xi$$

where H is a free (abelian) group of rank at most $|\mathfrak{J}_0|$.

The utility of Theorem 1.3 is that the computation of $G_1(R)$ for some hypersurface singularities becomes tractable, as well as removing the necessity of the injectivity of the Auslander-Reiten homomorphism required in [7]. However, before proving Theorem 1.3 in Section 1.3, we collect the necessary details on n -cluster tilting objects, noncommutative algebra and functor categories in Section 2.

Of course, in order to utilize Theorem 1.3, one might want to know when $\mathbf{mcm} R$ admits an n -cluster tilting object. This is discussed in Section 4.

The goal of explicitly computing $G_1(R)$ for specific R would not be possible if we could not compute $\text{Aut}_R(L)_{\text{ab}}$. We expend some energy in Section 5 calculating $\text{Aut}_R(L)_{\text{ab}}$ for several concrete examples.

Utilizing the results of Section 5, we are able to explicitly compute $G_1(R)$ for several hypersurface rings in Section 6

In Section 7, we discuss the similarities our examples share and make a conjecture.

We now fix notation. We always use A to denote an associative ring with identity that is not necessarily commutative; $\mathbf{mod} A$ will be the category of finitely generated left A -modules; and $\mathbf{proj} A$ will be the category of finitely generated projective left A -modules.

We will use the following setup: (R, \mathfrak{m}, k) always denotes a commutative local Cohen-Macaulay ring such that

- (a) R is Henselian.
- (b) R admits a dualizing module.
- (c) $\mathbf{mcm} R$ admits an n -cluster tilting object.
- (d) R is an isolated singularity.

The assumption of (a) give us that *any* maximal Cohen-Macaulay module can be written uniquely as a direct sum of finitely many indecomposable maximal Cohen-Macaulay modules (see ([11], Theorem 1.8) and Exercise 1.19). In fact, all of the rings for which we compute $G_1(R)$ are complete, so they already satisfy (a) (see ([11], Corollary 1.9)). The assumption of (b) is a standard technical assumption in representation theory of Cohen-Macaulay rings. Currently, the assumption (c) is very much a technical black box, but we will see it is indispensable. The assumption in (d) is necessary to make use of the theory of n -cluster tilting objects. When necessary, we will assume that R is a k -algebra and $\text{char}(k) \neq 2$, but we do not use this as a blanket assumption.

2. PRELIMINARIES

2.1. Some Definitions of K -groups. We begin first by discussing the classical definition lower K -groups.

Definition 2.1. The **classical K_0 -group** of A , denoted by $K_0^C(A)$, is defined as the Grothendieck group of the category $\mathbf{proj} A$. More explicitly, choose an isomorphism class for each $P \in \mathbf{proj} A$ and let X be the free abelian group on these isomorphism classes. Then $K_0^C(A)$ is the quotient of X by the subgroup of X generated by $\{[P] - [P'] - [P''] : 0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0 \text{ exact}\}$.

The **classical K_1 -group** of A , denoted by $K_1^C(A)$, is defined as the abelianization of the infinite general linear group over A . That is, using the obvious embeddings $GL_n(A) \hookrightarrow GL_{n+1}(A)$, we can form the *infinite general linear group* $GL(A) := \bigcup_{n \geq 1} GL_n(A)$. Thus $K_1^C(A)$ is $GL(A)_{\text{ab}}$.

Of principal importance in defining K -groups for our purposes is the following notion.

Definition 2.2. An **exact category** \mathcal{Y} is an additive category together with a distinguished class of sequences $Y' \rightarrowtail Y \twoheadrightarrow Y''$ called *conflations* with a fully faithful additive functor F from \mathcal{Y} into an abelian category \mathcal{X} such that

- (a) $Y' \rightarrowtail Y \twoheadrightarrow Y''$ is a conflation in \mathcal{Y} if and only if $0 \rightarrow F(Y') \rightarrow F(Y) \rightarrow F(Y'') \rightarrow 0$ is a short exact sequence in \mathcal{X} .
- (b) If $0 \rightarrow F(Y') \rightarrow X \rightarrow F(Y'') \rightarrow 0$ is exact in \mathcal{X} , then $X \cong F(Y)$ for some Y in \mathcal{Y} . That is, \mathcal{Y} is closed under extensions in \mathcal{X} .

We note any abelian category is an exact category. Moreover, $\mathbf{proj} A$ is an exact category, where the conflations are taken to be the sequences that are exact in $\mathbf{mod} A$. Note that $\mathbf{proj} A$ is an exact category which is not abelian.

We will need the following notions as they pertain to exact categories.

Definition 2.3. \mathcal{Y} denotes an exact category.

(a) We will always work under the assumption that the objects of \mathcal{Y} form a set. In this regard, we say that \mathcal{Y} is **skeletally small**.

(b) We say \mathcal{Y} is a **semisimple exact category** if every conflation splits. The prototypical example of a semisimple exact category is $\mathbf{proj} A$.

(c) We write \mathcal{Y}_0 to denote \mathcal{Y} viewed as an exact category in which the conflations $Y' \rightarrowtail Y \twoheadrightarrow Y''$ are such that the corresponding exact sequence in the abelian category \mathcal{X} is split exact. We call this the **trivial exact structure** for \mathcal{Y} .

The definition of Bass's K_1 functor rests squarely upon the following notion.

Definition 2.4. Let \mathcal{Y} be any category. Its **loop category** $\Omega\mathcal{Y}$ is the category whose objects are pairs (Y, α) , Y an object of \mathcal{Y} and $\alpha \in \text{Aut}_{\mathcal{Y}}(Y)$. A morphism in $\Omega\mathcal{Y}$ between two objects (Y, α) and (Y', α') is a commutative diagram in \mathcal{Y}

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ \alpha \downarrow \cong & & \cong \downarrow \alpha' \\ Y & \xrightarrow{f} & Y' \end{array}$$

Remark 2.5. Let \mathcal{Y} be a skeletally small exact category. Its loop category $\Omega\mathcal{Y}$ is also skeletally small and it is not hard to see that $\Omega\mathcal{Y}$ inherits an exact structure such that $(Y', \alpha') \rightarrowtail (Y, \alpha) \twoheadrightarrow (Y'', \alpha'')$ is a conflation in $\Omega\mathcal{Y}$ if and only if $Y' \rightarrowtail Y \twoheadrightarrow Y''$ is a conflation in \mathcal{Y} .

Definition 2.6. Let \mathcal{Y} be a skeletally small exact category and $\Omega\mathcal{Y}$ be its loop category, so that $\Omega\mathcal{Y}$ is also skeletally small and exact. We define **Bass's K_1 -group of \mathcal{Y}** , denoted by $K_1^B(\mathcal{Y})$, to be the Grothendieck group of $\Omega\mathcal{Y}$ modulo the subgroup generated by the following elements

$$(Y, \alpha) + (Y, \beta) - (Y, \alpha\beta)$$

For (Y, α) in $\Omega\mathcal{Y}$ we denote its image in $K_1^B(\mathcal{Y})$ as $[Y, \alpha]$.

Remark 2.7. (a) ([7], 3.4) We note for $Y \in \mathcal{Y}$, we have

$$[Y, 1_Y] + [Y, 1_Y] = [Y, 1_Y 1_Y] = [Y, 1_Y]$$

Hence $[Y, 1_Y]$ is the identity element of $K_1^B(\mathcal{Y})$.

(b) Unexpectedly, K_1^B is a functor from the category of skeletally small exact categories to abelian groups. Indeed, for a morphism F (which is necessarily an exact functor) between \mathcal{Y} and another skeletally small exact category, we have $K_1^B(F)([Y, \alpha]) = [F(Y), F(\alpha)]$.

Remark 2.8. ([16], Theorem 3.1.7 and) and ([7], Paragraph 3.5)

There is a natural isomorphism

$$\eta_A : K_1^C(A) \xrightarrow{\cong} K_1^B(\mathbf{proj} A)$$

The isomorphism η_A is such that $\xi \in GL_n(A)$ is mapped to the class $[A^n, \xi] \in K_1^B(\mathbf{proj} A)$, where elements of A^n are viewed as row vectors and ξ acts by multiplication on the right. The inverse η_A^{-1} acts as follows: Let $[P, \alpha]$ be in $K_1^B(\mathbf{proj} A)$ and choose P' in $\mathbf{proj} A$ and any isomorphism $P \oplus P' \rightarrow A^n$ of A -modules with $n \in \mathbb{N}$. In $K_1^B(\mathbf{proj} A)$, we have

$$[P, \alpha] = [P, \alpha] + [P', 1_{P'}]$$

There is an automorphism ϕ of A^n such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \longrightarrow & A^n & \longrightarrow & P' \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \phi & & \downarrow 1_{P'} \\ 0 & \longrightarrow & P & \longrightarrow & A^n & \longrightarrow & P' \longrightarrow 0 \end{array}$$

commutes in $\mathbf{proj} A$. Hence $0 \rightarrow (P, \alpha) \rightarrow (A^n, \phi) \rightarrow (P', 1_{P'}) \rightarrow 0$ is exact in $\Omega(\mathbf{proj} A)$. Thus

$$[P, \alpha] + [P', 1_{P'}] = [A^n, \phi]$$

The automorphism ϕ of (the row space) A^n can be identified with a matrix $\beta \in GL_n(A)$. The action of η_A^{-1} is then the image of β in $K_1^C(A)$.

Definition 2.9. Let \mathcal{Y} be a skeletally small exact category. The i th **Quillen K -group** of \mathcal{Y} , denoted by $K_i^Q(\mathcal{Y})$, is defined to be the abelian group $\pi_{i+1}(BQ\mathcal{Y}, 0)$, where $Q\mathcal{Y}$ is *Quillen's Q -construction*; $BQ\mathcal{Y}$ is the classifying space of $Q\mathcal{Y}$; 0 is a fixed zero object; and π_{i+1} denotes the taking of a homotopy group.

By ([15], Section 2, Theorem 1) there is a natural isomorphism of between the Grothendieck group functor and K_0^Q (as functors on the category of skeletally small exact categories). Moreover, $K_1^Q(\mathbf{proj} A)$ is naturally isomorphic to $K_1^C(A)$ (see ([18], Corollary 2.6 and Theorem 5.1)). Quillen's definition of higher K -theory is stunningly elegant, but does not often lend itself to performing computations with ease. The definition of Bass's functor K_1^B will be more suited for our computational needs and, we will want to exploit this in the sequel. As in ([7], 3.6), we will make strong use of the following theorem.

Theorem 2.10. *There exists a natural transformation $\zeta : K_1^B \rightarrow K_1^Q$, which we call the Gersten-Sherman transformation, of functors on the category of skeletally small exact categories such that $\zeta_{\mathcal{Y}} : K_1^B(\mathcal{Y}) \rightarrow K_1^Q(\mathcal{Y})$ is an isomorphism for every semisimple exact category \mathcal{Y} . In particular, $\zeta_{\mathbf{proj} A} : K_1^B(\mathbf{proj} A) \rightarrow K_1^Q(\mathbf{proj} A)$ is an isomorphism for every ring A .*

The name for ζ was introduced in [7] for the following: The existence of ζ was initially sketched by Gersten in ([6], sect. 5) and the details were later filled in by Sherman ([17], sect. 4), whom also proved $\zeta_{\mathcal{Y}}$ is an isomorphism for every semisimple exact category.

2.2. n -Auslander-Reiten Theory. We want to discuss generalizations of Auslander-Reiten theory, following [8]. To do so, we will require some precise categorical language. Here \mathcal{Y} denotes any exact category.

Definition 2.11. Write $\mathbf{Mod} \mathcal{Y}$ for the category of additive contravariant functors $\mathcal{Y} \rightarrow \mathbf{Ab}$, with \mathbf{Ab} the category of abelian groups. The morphisms in $\mathbf{Mod} \mathcal{Y}$ are natural transformations between functors with kernels and cokernels computed pointwise. An easy check shows that $\mathbf{Mod} \mathcal{Y}$ is abelian. We write (\bullet, Y) to denote the additive contravariant functor $\mathrm{Hom}_{\mathcal{Y}}(\bullet, Y)$. We say $F \in \mathbf{Mod} \mathcal{Y}$ is **finitely presented** if there is an exact sequence

$$(\bullet, Y) \rightarrow (\bullet, Y') \rightarrow F \rightarrow 0$$

in $\mathbf{Mod} \mathcal{Y}$. We write $\mathbf{mod} \mathcal{Y}$ for the subcategory of finitely presented functors.

For a ring A , let $\mathbf{Mod} A$ denote the category of all left A -modules and denote the subcategory of finitely presented left A -modules by $\mathbf{mod}_{\mathrm{fp}} A$. Fix a left A -module N and denote by E its endomorphism ring $\mathrm{End}_A(N)$. Then N has a left E -module structure that is compatible with its left A -module structure such that for $e \in E$ and $n \in N$, $e \cdot n = e(n)$. Denote by $\mathbf{add}_A N$ the category of A -modules that consists of all direct summands of finite direct sums of N . For $F \in \mathbf{Mod}(\mathbf{add}_A N)$, the aforementioned left E -module structure on N induces a left- E^{op} -module structure on the abelian group FN such that $e \cdot z = (Fe)(z)$ for $e \in E^{\mathrm{op}}$ and $z \in FN$. We use these facts for the following proposition, which will be essential in the proof of Theorem 1.3.

Proposition 2.12. ([7], Proposition 6.2)

There are quasi-inverse equivalences of abelian categories

$$\mathbf{Mod}(\mathbf{add}_A N) \begin{array}{c} \xrightarrow{e_N} \\ \xleftarrow{f_N} \end{array} \mathbf{Mod} E^{\mathrm{op}}$$

Where the functors e_N and f_N are defined as follows: $e_N(F) = FN$ (evaluation) and $f_N(Z) = Z \otimes_E \mathrm{Hom}_A(\bullet, N)|_{\mathbf{add}_A N}$ (functorification). Also, these quasi-inverse equivalences restrict to equivalences between categories of finitely presented objects

$$\mathbf{mod}(\mathbf{add}_A N) \begin{array}{c} \xrightarrow{e_N} \\ \xleftarrow{f_N} \end{array} \mathbf{mod}_{\mathrm{fp}} E^{\mathrm{op}}$$

Corollary 2.13. ([7], Observation 6.3) When $N = A$, $E = \text{End}_A(A) = A^{\text{op}}$, Proposition 2.12 yields an equivalence

$$f_A : \mathbf{mod}_{fp} A \longrightarrow \mathbf{mod}(\mathbf{proj} A)$$

Which is such that $X \longmapsto X \otimes_{A^{\text{op}}} \text{Hom}_A(\bullet, A)|_{\mathbf{proj} A}$. It is easy to see that the functor f_A is naturally isomorphic with the functor y_A , where $y_A(X) = \text{Hom}_A(\bullet, X)|_{\mathbf{proj} A}$. We identify the functor f_A with y_A .

Definition 2.14. The functor $y_N : \mathbf{add}_A N \longrightarrow \mathbf{mod}(\mathbf{add}_A N)$ given by $y_N(X) = \text{Hom}_A(\bullet, X)|_{\mathbf{add}_A N}$ is called the **Yoneda Functor**

Definition 2.15. Let \mathcal{X} be an additive category and \mathcal{C} a subcategory of \mathcal{X} . We call \mathcal{C} **contravariantly finite**, if for any $X \in \mathcal{X}$ there is a morphism $f : C \longrightarrow X$ with $C \in \mathcal{C}$ such that

$$(\bullet, C) \xrightarrow{\bullet f} (\bullet, X) \longrightarrow 0$$

is exact (where $\bullet f$ is the map induced by f). Such an f is called a **right- \mathcal{C} -approximation** of X . We dually define a **covariantly finite** subcategory and a **left- \mathcal{C} -approximation**. A contravariantly and covariantly finite subcategory is called **functorially finite**.

At long last, we are able to define an n -cluster tilting object.

Definition 2.16. Let \mathcal{Y} be an exact category with enough projectives. For objects X, Y in \mathcal{Y} we write $X \perp_n Y$ if $\text{Ext}_{\mathcal{Y}}^i(X, Y) = 0$ for $0 < i \leq n$. For an exact subcategory $\mathcal{C} \subset \mathcal{Y}$, we put

$$\mathcal{C}^{\perp_n} = \{X \in \mathcal{Y} : Y \perp_n X \text{ for all } Y \in \mathcal{C}\}$$

$${}^{\perp_n} \mathcal{C} = \{X \in \mathcal{Y} : X \perp_n Y \text{ for all } Y \in \mathcal{C}\}$$

We call \mathcal{C} an **n -cluster-tilting** subcategory of \mathcal{Y} if it is functorially finite and $\mathcal{C} = \mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}} \mathcal{C}$. An object L of \mathcal{Y} is called **n -cluster-tilting** if $\text{add}_{\mathcal{Y}}(L)$ is an n -cluster tilting subcategory.

From the definition of n -cluster tilting, if $\mathbf{mcm} R$ admits an n -cluster tilting object L , then R is necessarily a direct summand of L . While the definition of n -cluster tilting is quite a bit to digest at once, there are concrete examples of n -cluster tilting objects over familiar rings.

Example 2.17. (a) The motivating example of an n -cluster tilting comes from invariant theory. Let k be a field and S the ring $k[[x_1, \dots, x_n]]$. Suppose G is a finite subgroup of $GL_n(k)$ that does not contain any nontrivial pseudo-reflections and with $|G|$ invertible in k . Let R be the invariant subring $k[[x_1, \dots, x_n]]^G$ of S . If R is an isolated singularity, then the R -module S is an $(n-1)$ -cluster tilting object (see ([9], 2.5)).

(b) Let k be an algebraically closed field of characteristic not two and $S = k[[x, y]]$. Consider a reduced hypersurface ring $R = S/fS$. Then in [2] it is shown that if $f = f_1 \cdots f_n$ is a factorization of f into prime elements, $\mathbf{mcm} R$ has a 2-cluster tilting object if and only if $f_i \notin (x, y)^2$ for all i . In fact, it also shown in [2] that if $S_i = S/(f_1 \cdots f_i)S$ and $L = \bigoplus_{i=1}^n S_i$, then L is a 2-cluster tilting object for $\mathbf{mcm} R$.

When R has finite Cohen-Macaulay type, we have the classical notion of an *Auslander-Reiten sequence* or *almost-split sequence*. When $\mathbf{mcm} R$ has an n -cluster tilting subcategory, we have the following generalization.

Definition 2.18. If $\mathcal{C} \subset \mathbf{mcm} R$ is an n -cluster tilting subcategory, given $X \in \mathbf{mcm} R$ not free and indecomposable, an exact sequence

$$0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $C_0, \dots, C_n \in \mathcal{C}$ such that

$$0 \longrightarrow (\bullet, C_{n-1}) \xrightarrow{\bullet f_n} \cdots \xrightarrow{\bullet f_1} (\bullet, C_0) \xrightarrow{\bullet f_0} (\bullet, X) \longrightarrow 0$$

is a minimal projective resolution of $(\bullet, X)/\mathbf{rad}_{\mathbf{mcm} R}(\bullet, X)$ in $\mathbf{mod} \mathcal{C}$ is called an n -**Auslander-Reiten sequence** (or an n -**almost-split sequence**).

Here $\mathbf{rad}_{\mathbf{mcm} R}(\bullet, X)$ is such that

$$\mathbf{rad}_{\mathbf{mcm} R}(Y, X) = \{f \in \text{Hom}_R(Y, X) : fg \in \text{rad}(\text{End}_R(Y)) \text{ for all } g \in \text{Hom}_R(Y, X)\}$$

If $\mathcal{C} \subset \mathbf{mcm} R$ is an n -cluster tilting subcategory, then n -Auslander-Reiten sequences exist by ([9], Theorem 3.31).

2.3. Endomorphism Rings and K -groups. By our blanket assumptions on R , there is a unique decomposition of the n -cluster tilting object $L = L_1^{\oplus l_1} \oplus \cdots \oplus L_t^{\oplus l_t}$, such that $L_i \in \mathbf{mcm} R$ is indecomposable and $l_i > 0$ and the L_i are pairwise non-isomorphic. In this section, we will assume that $l_i = 1$. Indeed, this is immaterial, for if we write $L_{\text{red}} = L_1 \oplus \cdots \oplus L_t$, then $\mathbf{add}_R L = \mathbf{add}_R L_{\text{red}}$. Thus L is an n -cluster tilting object for $\mathbf{mcm} R$ if and only if L_{red} is. Write $\mathcal{C} = \mathbf{add}_R L$.

Construction. ([7], Construction 2.6)

If $L' \in \mathcal{C}$, we can write $L' = L_0^{\oplus l_0} \oplus \cdots \oplus L_t^{\oplus l_t}$ for uniquely determined $l_0, \dots, l_t \geq 0$. Set $q = q(L') = \max\{l_0, \dots, l_t\}$ and $v_j = v_j(L') = q - l_j$. Notice that q is the smallest integer such that L' is a direct summand of $L^{\oplus q}$. Now form the R -module $L'' = L_0^{\oplus v_0} \oplus \cdots \oplus L_t^{\oplus v_t}$ and let $\psi : L' \oplus L'' \rightarrow L^{\oplus q}$ be the R -linear isomorphism that takes the element

$$((\underline{x}_0, \dots, \underline{x}_t), (\underline{y}_0, \dots, \underline{y}_t)) \in L' \oplus L'' = (L_0^{\oplus l_0} \cdots \oplus L_t^{\oplus l_t}) \oplus (L_0^{\oplus v_0} \oplus \cdots \oplus L_t^{\oplus v_t})$$

where $\underline{x}_j \in L^{\oplus l_j}$ and $\underline{y}_j \in L_j^{\oplus l_j}$, to the element

$$((z_{01}, \dots, z_{t1}), \dots, (z_{0q}, \dots, z_{tq})) \in L^{\oplus q} = (L_0 \oplus \cdots \oplus L_t)^{\oplus q}$$

with $z_{j1}, \dots, z_{jq} \in L_j$ given by

$$(z_{j1}, \dots, z_{jq}) = (\underline{x}_j, \underline{y}_j) \in L_j^{\oplus q} = L_j^{\oplus (l_j + v_j)}$$

Now for $\alpha \in \text{Aut}_R(L')$, we define $\tilde{\alpha}$ to be the automorphism on $L^{\oplus q}$ given by $\psi(\alpha \oplus 1_{L''})\psi^{-1}$. Note that $\tilde{\alpha} = (\widetilde{\alpha_{ij}})$, with $\widetilde{\alpha_{ij}}$ uniquely determined endomorphisms of L . In particular, $\tilde{\alpha} \in \mathbb{M}_q(\text{End}_R L)$. As in [7], we refer to this construction as the **tilde construction**.

Remark 2.19. We note a special case of the tilde construction. Suppose $\alpha = a1_{L'}$ with $a \in R^*$. If $L' = L_{i_1}^{\oplus q} \oplus \cdots \oplus L_{i_h}^{\oplus q}$ with $0 \leq i_1 < i_2 < \cdots < i_h \leq t$. Then $\tilde{\alpha} : L^{\oplus q} \rightarrow L^{\oplus q}$ is the automorphism given by $e1_{L^{\oplus q}}$ with $e \in \text{Aut}_R(L)$ given by

$$\text{diag}(1_{L_0}, \dots, a1_{L_{i_1}}, \dots, a1_{L_{i_h}}, \dots, 1_{L_t})$$

Hence, $(\widetilde{a1_{L'}})^{-1} = \widetilde{a^{-1}1_{L'}}$.

Our principal reason for utilizing the tilde construction is the following. Keep notation as above.

Lemma 2.20. ([7], Lemma 8.7).

Let $\Lambda = \text{End}_R(L)^{\text{op}}$. By Remark 2.8, there is an isomorphism $\eta_\Lambda : K_1^C(\Lambda) \rightarrow K_1^B(\mathbf{proj} \Lambda)$. For L' in \mathcal{C} and $\alpha \in \text{Aut}_R(L')$, consider the element

$$\xi_{L', \alpha} = [\text{Hom}_R(L, L'), \text{Hom}_R(L, \alpha)] \in K_1^B(\mathbf{proj} \Lambda)$$

Let $\tilde{\alpha} \in GL_q(\Lambda)$ be given by the tilde construction applied to α . Then $\eta_\Lambda(\tilde{\alpha}^T) = \xi_{L', \alpha}$.

As we will often be working explicitly with highly noncommutative rings, we need to discuss important ideas at the intersection of noncommutative algebra and K -theory. Let J be the Jacobson radical of the not necessarily commutative ring A . Recall that A is said to be *semilocal* if A/J is semisimple. That is, every left A/J -module has the property that each of its submodules is a direct summand of A/J . In the case that A is commutative, this is equivalent to A having only finitely many maximal ideals ([10], Proposition 20.2). Of great importance to us is the following situation: If A is a commutative semilocal Noetherian ring and N is a nonzero finitely generated A -module, then $\text{End}_A(N)$ is semilocal in the preceding sense ([7], Lemma 5.1). Without assuming A is commutative

or the N_j are finitely generated, we show that $\text{End}_A(N)$ is semilocal if it is a finite direct sum of modules whose endomorphism rings are local in Proposition 5.1. Either route shows that the $\text{End}_R(M)$ is semilocal and we will see how the following remark utilizes this small but essential fact in the proof of Theorem 1.3.

Remark 2.21. ([7], Paragraph 5.2)

For arbitrary A , denote the composition of the following group homomorphisms

$$A^* = GL_1(A) \hookrightarrow GL(A) \twoheadrightarrow GL(A)_{\text{ab}} = K_1^C(A)$$

by ϑ_A . Since $K_1^C(A)$ is abelian, there is an induced map $\theta_A : A_{\text{ab}}^* \rightarrow K_1^C(A)$. If A is semilocal, then ([1], V§9 Theorem 9.1) shows that ϑ_A is surjective, hence so is θ_A . When A contains a field k with $\text{char}(k) \neq 2$, a result of Vaserstein ([20], Theorem 2) shows that θ_A is an isomorphism. In particular, if R is a k -algebra, $\text{char}(k) \neq 2$ and M is a finitely generated R -module with $E = \text{End}_R(M)$, then θ_E and $\theta_{E^{\text{op}}}$ are isomorphisms.

Suppose now A is a commutative semilocal ring, so that the commutator subgroup $[A^*, A^*]$, is trivial, hence $\theta_A : A^* \rightarrow K_1^C(A)$ is surjective. In ([7], Remark. 5.4), an explicit inverse to θ_A is constructed: The determinant homomorphisms $\det_n : GL_n(A) \rightarrow A^*$ induce a homomorphism $\det_A : K_1^C(A) \rightarrow A^*$ (since each \det_n is trivial on commutators in $GL(A)$) which satisfies $\det_A \theta_A = 1_{A^*}$, so that $\theta_A^{-1} = \det_A$.

Using Remark 2.21 as motivation, the following definition is made in [7].

Definition 2.22. Let A be a ring for which the map $\theta_A : A_{\text{ab}}^* \rightarrow K_1^C(A)$ is an isomorphism. The inverse θ_A^{-1} is denoted by \det_A and is called the **generalized determinant**.

Remark 2.23. ([7], Observation 8.9)

Let A be any commutative Noetherian local ring and η_A be the isomorphism from Remark 2.8 and $\theta_A : A^* \rightarrow K_1^C(A)$ be the induced map from Remark 2.21. Then θ_A is an isomorphism by ([18], Example 1.6). Thus the composition $\rho_A = \eta_A \theta_A : A^* \rightarrow K_1^B(\mathbf{proj} A)$ is an isomorphism such that $a \in A^*$ is mapped to $[A, a1_A]$.

3. THE STRUCTURE OF $G_1(R)$

In this section, unadorned K -groups are the Quillen K -groups. Our goal of this section is to prove Theorem 1.3. We always assume that $\mathbf{mcm} R$ has an n -cluster tilting object $L = L_0^{l_0} \oplus \cdots \oplus L_t^{\oplus l_t}$ such that $\Lambda := \text{End}_R(L)^{\text{op}}$ has finite global dimension. We start with an easy reduction.

Lemma 3.1. Set $L_{\text{red}} = L_0 \oplus \cdots \oplus L_t$. If $\Lambda_{\text{red}} = \text{End}_R(L_{\text{red}})^{\text{op}}$, then Λ and Λ_{red} are Morita-equivalent. In particular, $G_i(\Lambda) \cong G_i(\Lambda_{\text{red}})$ for all $i \geq 0$.

Proof. The desired Morita equivalence is from ([4], Lemma 2.2). Thus the categories of left Λ and Λ_{red} modules are equivalent, hence there is an equivalence of exact categories between $\mathbf{mod} \Lambda$ and $\mathbf{mod} \Lambda_{\text{red}}$. It is well-known this yields an isomorphism in G -theory, hence $G_i(\Lambda) \cong G_i(\Lambda_{\text{red}})$ for all $i \geq 0$. □

As previously noted, $\mathbf{add}_R L = \mathbf{add}_R L_{\text{red}}$, so that with Lemma 3.1 in hand, we may safely assume that the n -cluster tilting object L for $\mathbf{mcm} R$ has the form $L_0 \oplus \cdots \oplus L_t$, where the L_i are non-isomorphic indecomposable maximal Cohen-Macaulay. Moreover, $\text{End}_R(L_0 \oplus \cdots \oplus L_t)^{\text{op}}$ has finite global dimension. We always use Λ to denote $\text{End}_R(L)^{\text{op}}$. We need the following.

Lemma 3.2. There is an isomorphism $G_0(\Lambda) \cong \mathbb{Z}^{\oplus(t+1)}$.

Proof. By Proposition 2.12, there is a quasi-inverse equivalence of categories $e_L : \mathbf{mod}(\mathbf{add}_R L) \rightarrow \mathbf{mod} \Lambda$, which gives an isomorphism $K_0(\mathbf{mod}(\mathbf{add}_R L)) \cong K_0(\mathbf{mod} \Lambda) = G_0(\Lambda)$. Since Λ has finite global dimension, there is an isomorphism $K_0((\mathbf{add}_R L)_0) \cong K_0(\mathbf{mod}(\mathbf{add}_R L))$ by ([7], Lemma 6.5). Now ([7], Lemma 6.6) gives that $K_0((\mathbf{add}_R L)_0) \cong \mathbb{Z}^{\oplus(t+1)}$, since $\mathbf{mod} R$ is Krull-Schmidt. □

We now assume k is algebraically closed. In this case, $k = \kappa_{L_j}$ for all j (this is essentially Nakayamma's lemma; see (c) of Proposition 5.1 for details). As k has finite global dimension, $G_1(k) = K_1(k) = k^*$ and $G_0(k) = K_0(k) = \mathbb{Z}$. Thus by Theorem 1.2, there is an exact sequence of abelian groups

$$(1) \quad (k^*)^{\oplus t} \longrightarrow G_1(\Lambda) \longrightarrow G_1(R) \longrightarrow G_0(\Lambda) \longrightarrow G_0(R) \longrightarrow 0$$

Using Lemma 3.2, we have an exact sequence

$$(\star) \quad (k^*)^{\oplus t} \longrightarrow G_1(\Lambda) \longrightarrow G_1(R) \longrightarrow H \longrightarrow 0$$

Where H is the kernel of a map $\mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$. From (\star) , it is obvious that H is a direct summand of $G_1(R)$, since it is free of rank at most t . We derive useful information about the structure of $G_1(\Lambda)$ for the purpose of proving Theorem 1.3.

Lemma 3.3. *Let A be of finite global dimension. There is an isomorphism of abelian groups*

$$G_1(A) \cong K_1(A) \cong K_1^B(\mathbf{proj} A) \cong K_1^C(A)$$

Proof. The assumption that A has finite global dimension allows us to apply Quillen's Resolution Theorem ([15], §Theorem 3) to give the first isomorphism. The second arises from the Gersten-Sherman transformation via Theorem 2.10. The last is the isomorphism is from Remark 2.8 \square

Lemma 3.4. *Suppose that A is semilocal and contains a field that has characteristic not two. If A^{op} has finite global dimension, then*

$$G_1(A^{op}) \cong A_{ab}^*$$

Proof. By ([18], V§7), transposition $(\bullet)^T : GL_n(A^{op}) \longrightarrow GL_n(A)$ is an anti-isomorphism, hence there is a induced isomorphism $(\bullet)^T : K_1^C(A^{op}) \longrightarrow K_1^C(A)$. By ([20], Theorem 2), $K_1^C(A) \cong A_{ab}^*$. The statement now follows from Lemma 3.3 by replacing A with A^{op} . \square

Now our assumption is that Λ has finite global dimension, hence the exact sequence (\star) becomes

$$(\star\star) \quad (k^*)^{\oplus t} \xrightarrow{\gamma} \Lambda_{ab}^* \longrightarrow G_1(R) \longrightarrow H \longrightarrow 0$$

In particular, there is an isomorphism

$$G_1(R) \cong \text{coker}(\gamma) \oplus H$$

In order to prove Theorem 1.3, we need describe the map $\gamma : (k^*)^{\oplus t} \longrightarrow \Lambda_{ab}^*$. With $L = L_0 \oplus \cdots \oplus L_t$, we assume that $L_0 = R$ and L_1, \dots, L_t are the non-free indecomposable and pairwise non-isomorphic summands of L . We set $\mathfrak{J} = \{L_0, \dots, L_t\}$ and $\mathfrak{J}_0 = \mathfrak{J} \setminus \{R\}$. Now set $\mathcal{C} = \mathbf{add}_R L$ and let \mathcal{C}_0 be the category \mathcal{C} equipped with the trivial exact structure (see Definition 2.3). We start from afar by using the description of a map $\alpha : \bigsqcup_{\mathfrak{J}_0} K(\mathbf{mod} k) \longrightarrow K(\mathcal{C}_0)$ from ([13], Section 7.2)

that is used to construct the long exact sequence of Theorem 1.2. For $j > 0$ let

$$0 \longrightarrow C_n^j \longrightarrow \cdots \longrightarrow C_0^j \longrightarrow L_j \longrightarrow 0$$

be the n -Auslander-Reiten sequence ending in L_j (see Definition 2.18). Denote by k_j the object of $\oplus_{\mathfrak{J}_0} \mathbf{mod} k$ which is k in the L_j -coordinate and 0 in the others. We remark that to define a k -linear functor out of $\oplus_{\mathfrak{J}_0} \mathbf{mod} k$, one needs only to specify the image of each object k_j . We define k -linear functors

$$a_i : \bigoplus_{\mathfrak{J}_0} \mathbf{mod} k \longrightarrow \mathcal{C}_0 \quad (0 \leq i \leq n+1)$$

by

$$\begin{cases} a_i(k_j) = C_{i-1}^j & (1 \leq i \leq n+1) \\ a_0(k_j) = L_j \end{cases}$$

It is shown in ([13], Section 7.2) that $\alpha = \sum_{i=0}^{n+1} (-1)^i K(a_i)$. We simplify notation for the proof and write $\bigoplus_{\mathcal{I}_0} \mathbf{mod} k = (\mathbf{mod} k)^{\oplus t}$ and abuse notation to write α for the corresponding map on Quillen K_1 -groups from $(\mathbf{mod} k)^{\oplus t}$ to \mathcal{C}_0 . We now have the tools to prove Theorem 1.3.

Proof. Let $\beta = \sum_{i=0}^{n+1} (-1)^i K_1^B(a_i)$ be the corresponding map on the Bass K_1 -groups. As the functors K_1 and K_1^B are additive, we have $K_1((\mathbf{mod} k)^{\oplus t}) = K_1(\mathbf{mod} k)^{\oplus t}$ and $K_1^B((\mathbf{mod} k)^{\oplus t}) = K_1^B(\mathbf{mod} k)^{\oplus t}$. Now the Gersten-Sherman transformation ζ provides the following commutative diagram for $i = 0, 1, \dots, n+1$

$$\begin{array}{ccc} K_1(\mathbf{mod} k)^{\oplus t} & \xrightarrow{K_1(a_i)} & K_1(\mathcal{C}_0) \\ \oplus_t \zeta_{\mathbf{mod} k} \downarrow \cong & & \cong \downarrow \zeta_{\mathcal{C}_0} \\ K_1^B(\mathbf{mod} k)^{\oplus t} & \xrightarrow{K_1^B(a_i)} & K_1^B(\mathcal{C}_0) \end{array}$$

Where the vertical isomorphisms come courtesy of Theorem 2.10, as \mathcal{C}_0 and $\mathbf{mod} k$ are semisimple exact categories. Hence there is a commutative diagram

$$\begin{array}{ccc} K_1(\mathbf{mod} k)^{\oplus t} & \xrightarrow{\alpha} & K_1(\mathcal{C}_0) \\ \oplus_t \zeta_{\mathbf{mod} k} \downarrow \cong & & \cong \downarrow \zeta_{\mathcal{C}_0} \\ K_1^B(\mathbf{mod} k)^{\oplus t} & \xrightarrow{\beta} & K_1^B(\mathcal{C}_0) \end{array}$$

Thus we may identify α with β .

Recall the category $\mathbf{mod} \mathcal{C}$ of finitely presented additive contravariant functors on \mathcal{C} . We have the exact Yoneda functor $y_L : \mathcal{C}_0 \rightarrow \mathbf{mod} \mathcal{C}$ given by $y_L(X) = \mathrm{Hom}_R(\bullet, X)|_{\mathcal{C}}$ (recall $\mathcal{C} = \mathbf{add}_R(L)$), so this coincides with Definition 2.14). By ([7], Lemma 6.5), $K_1^B(y_L) : K_1^B(\mathcal{C}_0) \rightarrow K_1^B(\mathbf{mod} \mathcal{C})$ is an isomorphism. Also recall the quasi-inverse equivalence $e_L : \mathbf{mod} \mathcal{C} \rightarrow \mathbf{mod} \Lambda$ given by evaluation at L from Proposition 2.12. Since e_L is a quasi-inverse equivalence, there is an induced isomorphism $K_1^B(e_L) : K_1^B(\mathbf{mod} \mathcal{C}) \rightarrow K_1^B(\mathbf{mod} \Lambda)$. Now define $\Gamma : (k^*)^{\oplus t} \rightarrow K_1^B(\mathbf{mod} \Lambda)$ as follows: Let $a \in k^*$ be the j th component of an element of $(k^*)^{\oplus t}$, then Γ maps a to the element $\xi_{j,a}$ of $K_1^B(\mathbf{mod} \Lambda)$ given by

$$\xi_{j,a} = [\mathrm{Hom}_R(L, L_j), \mathrm{Hom}_R(L, a1_{L_j})] + \sum_{i=0}^n (-1)^{i+1} [\mathrm{Hom}_R(L, C_i^j), \mathrm{Hom}_R(L, a1_{C_i^j})]$$

Notice Γ is a group homomorphism by the relations that define $K_1^B(\mathbf{mod} \Lambda)$. Recall the isomorphism $\rho_k : k^* \rightarrow K_1^B(\mathbf{proj} k)$ from Remark 2.23. The map ρ_k sends $a \in k^*$ to $[k, a1_k]$. Since $\mathbf{mod} k = \mathbf{proj} k$, we have the following commutative diagram

$$\begin{array}{ccc} K_1^B(\mathbf{proj} k)^{\oplus t} & \xrightarrow{\beta} & K_1^B(\mathcal{C}_0) \\ \oplus_t \rho_k^{-1} \downarrow \cong & & \cong \downarrow K_1^B(e_L y_L) \\ (k^*)^{\oplus t} & \xrightarrow{\Gamma} & K_1^B(\mathbf{mod} \Lambda) \end{array}$$

Indeed, commutativity of the above diagram can be seen by checking on the elements $[k, a1_k] \in K_1^B(\mathbf{proj} k)^{\oplus t}$ in the j th position. Hence the homomorphisms β and Γ can be identified.

Lastly, let σ be the composition of the following isomorphisms

$$K_1^B(i)^{-1} : K_1^B(\mathbf{mod} \Lambda) \rightarrow K_1^B(\mathbf{proj} \Lambda)$$

$$\eta_\Lambda^{-1} : K_1^B(\mathbf{proj} \Lambda) \longrightarrow K_1^C(\Lambda)$$

$$\det_\Lambda : K_1^C(\Lambda) \longrightarrow \Lambda_{\text{ab}}^*$$

$K_1^B(i)$ is the map induced by the inclusion $i : \mathbf{proj} \Lambda \longrightarrow \mathbf{mod} \Lambda$ and is an isomorphism by Bass's Resolution Theorem ([1], VIII§4 Theorem 4.6); η_Λ is the isomorphism of Remark 2.8; \det_Λ is the isomorphism discussed in Remark 2.21. Then $\sigma\Gamma : (k^*)^{\oplus t} \longrightarrow \Lambda_{\text{ab}}^*$ is the map γ , so that $\text{coker}(\gamma) = \text{coker}(\sigma\Gamma)$. We now compute $\text{im}(\sigma\Gamma)$. Since $\xi_{j,a}$ is already an element of $K_1^B(\mathbf{proj} \Lambda)$, we have

$$\sigma\Gamma(a) = \sigma(\xi_{j,a}) = \det_\Lambda \eta_\Lambda^{-1}(\xi_{j,a})$$

Using Remark 2.19 with $q = 1$, Lemma 2.20 and ([7], Lemma 5.7), we have

$$\det_\Lambda \eta_\Lambda^{-1}(\xi_{j,a}) = \widetilde{a1_{L_j}} \prod_{i=0}^n \det_{\Lambda^{\text{op}}}(\widetilde{a1_{C_i^j}})^{(-1)^{i+1}}$$

Where $\det_{\Lambda^{\text{op}}}$ is the isomorphism of Remark 2.21. This gives Theorem 1.3 with Ξ the subgroup generated by $\widetilde{a1_{L_j}} \prod_{i=0}^n \det_{\Lambda^{\text{op}}}(\widetilde{a1_{C_i^j}})^{(-1)^{i+1}}$ ($a \in k^*$).

□

4. EXISTENCE OF n -CLUSTER TILTING OBJECTS IN $\mathbf{mcm} R$

Naturally, the usefulness of Theorem 1.3 would be limited if the situations in which $\mathbf{mcm} R$ contained an n -cluster tilting object were sparse. Fortunately for us, they are not. Moreover, if $\mathbf{mcm} R$ admits an n -cluster tilting object L , we require that $\Lambda := \text{End}_R(L)^{\text{op}}$ has finite global dimension. At first glance, this condition might also seem limiting, but is in fact quite common, as seen in the following theorem.

Theorem 4.1. ([8], Theorem 3.12(a))

Suppose $\dim R = d$ and that $\mathbf{mcm} R$ contains an n -cluster tilting object L with $d \leq n$. Then Λ has global dimension at most $n + 1$.

The most well-studied situation in which $\mathbf{mcm} R$ admits an n -cluster tilting object is the following.

4.1. Finite Cohen-Macaulay Type. Recall that we say that R has finite Cohen-Macaulay type (or *finite type* for short) when R has only finitely many indecomposable maximal Cohen-Macaulay modules. Now the only 1-cluster tilting subcategory of $\mathbf{mcm} R$ is $\mathbf{mcm} R$ itself. Thus the existence of a 1-cluster tilting object for $\mathbf{mcm} R$ is equivalent to R having finite type. In particular, when R has finite type, $\mathbf{mcm} R$ has an additive generator M . For practical and computational purposes, when R has finite type, we will often work with the R -module $M = M_0 \oplus M_1 \oplus \cdots \oplus M_t$, with $M_0 = R$ and M_1, \dots, M_t the pairwise non-isomorphic and non-free indecomposable maximal Cohen-Macaulay R -modules. Moreover, ([12], Theorem 2.1) shows that $\text{End}_R(M)^{\text{op}}$ has finite global dimension, hence Theorem 1.3 is applicable in this situation.

4.1.1. ADE Singularities. The most important examples of rings that have finite type are the simple surface singularities. These are called the ADE singularities. Let $S = k[[x, y, z_2, z_3, \dots, z_d]]$ and assume k is algebraically closed with characteristic different from 2, 3 and 5. Set $R = S/fS$ with f nonzero and $f \notin (x, y, z_2, \dots, z_d)^2$. The f for which R has finite type are exactly the following ([11], Theorem 9.8)

$$\begin{array}{lll} (A_n) & x^2 + y^{n+1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (n \geq 1) \\ (D_n) & x^2 y + y^{n-1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (n \geq 4) \\ (E_6) & x^3 + y^4 + z_2^2 + z_3^2 + \cdots + z_d^2 & \\ (E_7) & x^3 + xy^3 + z_2^2 + z_3^2 + \cdots + z_d^2 & \\ (E_8) & x^3 + y^5 + z_2^2 + z_3^2 + \cdots + z_d^2 & \end{array}$$

4.2. Invariant Subrings. Let k be a field and S the ring $k[[x_1, \dots, x_n]]$. Suppose G is a finite subgroup of $GL_n(k)$ that does not contain any nontrivial pseudo-reflections and with $|G|$ invertible in k . Let R be the invariant subring $k[[x_1, \dots, x_n]]^G$ of S , where G acts by a linear change of variables on S . If R is an isolated singularity, then the R -module S is an $(n-1)$ -cluster tilting object (see ([9], 2.5)).

The skew group ring of S , denoted by $S\#G$ is such that, as an S -module is given by $S\#G = \bigoplus_{\sigma \in G} S \cdot \sigma$ and has multiplication given by $(s \cdot \sigma)(t \cdot \tau) = s\sigma(t) \cdot \sigma\tau$. In this situation, $S\#G$ has global dimension equal to n ([11], Corollary 5.8) and there is an isomorphism $\text{End}_R(S) \cong S\#G$ ([11], Theorem 5.15). In particular, Theorem 1.3 is applicable in this situation.

4.3. Reduced Hypersurface Singularities.

4.3.1. Dimension One. Let k be an algebraically closed field of characteristic not two and $S = k[[x, y]]$. For $f \in (x, y)$, let $R = S/fS$ be a reduced hypersurface. Suppose f has prime factorization and $f = f_1 \cdots f_n$, $S_i = S/(f_1 \cdots f_i)S$ and L is the R -module $S_1 \oplus \cdots \oplus S_n$. If $f_i \notin (x, y)^2$ for all i , then [2] shows that L is a 2-cluster tilting object for $\mathbf{mcm} R$. Moreover, Theorem 4.1 shows that $\text{End}_R(L)^{\text{op}}$ has global dimension at most three. Hence we can apply Theorem 1.3 in this situation. Note, in particular, if $\lambda_1, \dots, \lambda_n$ are distinct elements of k , then Theorem 1.3 is applicable to the ring S/fS with $f = (x - \lambda_1 y) \cdots (x - \lambda_n y)$.

4.3.2. Dimension Three. Keep notation as in the previous section. Set $S' = k[[x, y, u, v]]$ and $R' = S'/(f + uv)S'$. Then $\mathbf{mcm} R'$ has a 2-cluster tilting object if $f_i \notin (x, y)^2$ for all i and it is given by $L := U_1 \oplus \cdots \oplus U_n$ with $U_i = (u, f_1 \cdots f_i) \subset R'$ ([8], Theorem 4.17). Moreover, ([8], Theorem 4.17) also says $\text{End}_{R'}(L)^{\text{op}}$ has finite global dimension, so Theorem 1.3.

5. ABELIANIZATION OF AUTOMORPHISM GROUPS

Of course, the usefulness of Theorem 1.3 would be limited if one were unable to compute $\text{Aut}_R(L)_{\text{ab}}$. While we do have some results that fit into a general framework, most of our computations are ad hoc and tailored specifically to each ring. Our computations rely significantly upon the general framework laid out by [7] and this work serves strongly as inspiration for our results.

We set up some useful notation. Let N_1, \dots, N_s be A -modules and consider the A -module $N := N_1 \oplus \cdots \oplus N_s$. We view the elements of N as column vectors and the endomorphism ring of N has a matrix-like structure: For $f \in \text{End}_A(N)$, we can write $f = (f_{ij})$ with $f_{ij} \in \text{Hom}_A(N_j, N_i)$ and composition with another endomorphism $g = (g_{ij})$ can be accomplished in the same manner one would multiply matrices with entries in A . We write a $\text{diag}(\alpha_1, \dots, \alpha_s)$ for the diagonal endomorphism of N with $\alpha_i \in \text{End}_A(N_i)$. For $\alpha \in \text{Aut}_A(N_j)$, we denote by $d_j(\alpha)$ the automorphism of N given by $\text{diag}(1_{N_1}, \dots, 1_{N_{j-1}}, \alpha, 1_{N_{j+1}}, \dots, 1_{N_s})$. For $i \neq j$ and $\beta \in \text{Hom}_A(N_j, N_i)$, we denote by $e_{ij}(\beta)$ the automorphism of N with diagonal entries $1_{N_1}, \dots, 1_{N_s}$ and (i, j) th entry given by β and zeros elsewhere.

Over our ring R , each indecomposable maximal Cohen-Macaulay module has a local endomorphism ring and the R -module M is a finite direct sum of these modules, so it would be most pertinent to investigate the structure of the endomorphism ring of a module that is a finite direct sum of modules whose endomorphism rings are local. We accomplish this in the following proposition.

Proposition 5.1. *Let N_1, \dots, N_s be A -modules and $N = N_1 \oplus \cdots \oplus N_s$. Suppose each N_j has a local endomorphism ring and the N_j are pairwise nonisomorphic. Let E_j denote the endomorphism ring of N_j , \mathfrak{m}_j the Jacobson radical of E_j and D_j the division ring E_j/\mathfrak{m}_j . Let E be the endomorphism ring of N and J its Jacobson radical. Then*

- (a) $J = \{(\alpha_{ij}) : \alpha_{ij} \in \mathfrak{m}_j\}$ and $E/J \cong D_1 \times \cdots \times D_s$. In particular, E is semilocal.
- (b) There is a surjection

$$E_{ab}^* \twoheadrightarrow (D_1^*)_{ab} \times \cdots \times (D_s^*)_{ab}$$

In case D_j is a field for all j , this surjection is such that when $n = 2$ and 2 is a unit in A or $n \geq 3$, the kernel of this map consists of the images of $\text{diag}(\alpha_1, \dots, \alpha_s) \in E$ such that $\alpha_{jj} \in 1_{N_j} + \mathfrak{m}_j \in \text{Aut}_A(N_j)$

(c) If k is a field with $k \subseteq A$ and $k = D_j$ for all j , then $(k^*)^{\oplus s}$ is a direct summand of E_{ab}^* . Now suppose k is algebraically closed and A is commutative and local with maximal ideal \mathfrak{n} , so that $A/\mathfrak{n} = k$. If the N_j are finitely generated, then $k = D_j$ for all j , so that E_{ab}^* contains $(k^*)^{\oplus s}$ as a direct summand.

Proof. (a) Suppose we have a left-invertible element $\alpha = (\alpha_{ij})$ of E , where $\alpha_{ij} \in \text{Hom}_A(N_j, N_i)$ and $\beta = (\beta_{ij})$ is its left inverse. We have

$$1_{N_j} = (\beta\alpha)_{jj} = \sum_{k=1}^s \beta_{jk}\alpha_{kj}$$

By ([7], Lemma 9.3), when $j \neq k$, $\beta_{jk}\alpha_{kj} \in \mathfrak{m}_j$. Thus $\beta_{jj}\alpha_{jj}$ is left-invertible in E_j , hence α_{jj} is left-invertible in E_j . Suppose, conversely, that $\alpha = (\alpha_{ij})$ is such that each α_{jj} is left-invertible in E_j with left inverse β_j . Let β be the diagonal automorphism with entries β_1, \dots, β_s . Then $(\beta\alpha)_{jj} = 1_{N_j}$ and by ([7], Proposition 9.4), $\beta\alpha$ is invertible, thus α is left-invertible. In particular, we have shown that $\alpha = (\alpha_{ij})$ is left-invertible in E if and only if α_{jj} is left-invertible in E_j for all j .

Now let $\alpha = (\alpha_{ij}) \in E$ with $\alpha_{jj} \in \mathfrak{m}_j$ for all j . For $\beta = (\beta_{ij}) \in E$, we have

$$(1_N - \beta\alpha)_{jj} = 1_{N_j} - \sum_{k=1}^s \beta_{jk}\alpha_{kj} \in E_j$$

As before, when $j \neq k$, $\beta_{jk}\alpha_{kj} \in \mathfrak{m}_j$ by ([7], Lemma 9.3). Since $\alpha_{jj} \in \mathfrak{m}_j$, the right-hand side of the above equality is invertible in E_j . By ([7], Proposition 9.4), we have that $1_N - \beta\alpha$ is invertible, hence $\alpha \in J$. Conversely, for $\alpha \in J$, $1_N - \beta\alpha$ is left-invertible for all $\beta \in E$. If $\beta_j(\gamma)$ is the endomorphism of N with $\gamma \in E_j$ in the j th diagonal position and zeroes elsewhere, then $1_N - \beta_j(\gamma)\alpha$ is left-invertible, hence by the preceding paragraph, $1_{N_j} - \gamma\alpha_{jj}$ is left-invertible in E_j , so that $\alpha_{jj} \in \mathfrak{m}_j$. Thus we have shown $\alpha = (\alpha_{ij}) \in J$ if and only if $\alpha_{jj} \in \mathfrak{m}_j$ for all j .

The above shows that every element of E/J can be represented by a diagonal endomorphism. Hence for $\alpha, \beta \in E$ with $(\alpha - \beta)_{jj} \in \mathfrak{m}_j$, we have $\alpha - \beta \in J$. This gives a well-defined ring homomorphism $E/J \rightarrow D_1 \times \dots \times D_s$ given by $(\alpha_{ij}) \mapsto (\alpha_{11} + \mathfrak{m}_1, \dots, \alpha_{ss} + \mathfrak{m}_s)$. This map is easily seen to be an isomorphism.

(b) By ([7], Proposition 9.4), $(\alpha_{ij}) \in E$ is invertible if and only if $\alpha_{jj} \notin \mathfrak{m}_j$ for all j . In particular, if $(\alpha_{ij}) \in E^*$, then its image in E/J is invertible. This gives a surjection

$$E^* \twoheadrightarrow (E/J)^*$$

Using (a) and the right-exactness $(\bullet)_{ab}$, there is an induced surjection

$$E_{ab}^* \twoheadrightarrow (D_1^*)_{ab} \times \dots \times (D_s^*)_{ab}$$

If D_j is a field, then $D_j^* = (D_j^*)_{ab}$. If $n = 2$ and 2 is invertible in A or $n \geq 3$, we may appeal to ([7], Corollary 9.5) to see every element of E_{ab}^* can be represented by a diagonal endomorphism and it is easy to see that the kernel consists precisely of the elements given in the statement.

(c) Since $k \subseteq A$ there is an induced map $\varphi : (k^*)^{\oplus s} \rightarrow E_{ab}^*$ which takes (a_1, \dots, a_s) to the image of $\text{diag}(a_1 1_{M_1}, \dots, a_s 1_{M_s})$ in E_{ab}^* . Since $k = D_j$ for all j , we have $(E/J)_{ab}^* = (k^*)^{\oplus s}$. The map in (b) splits φ , hence $(k^*)^{\oplus s}$ is a direct summand of E_{ab}^* .

For the second part, Nakayama's lemma shows that $\mathfrak{n}E_j \subseteq \mathfrak{m}_j$. Thus $D_j = E_j/\mathfrak{m}_j$ is a vector space over $k = A/\mathfrak{n}$. Since each N_j is finitely generated, D_j is a finite-dimensional division algebra over k , we have $k = D_j$.

□

We now compute E_{ab}^* over specified R for explicit M . In all of the examples chosen, we will see the R -module M is an n -cluster tilting object in $\mathbf{mcm} R$ and in some cases, is in an additive generator for $\mathbf{mcm} R$ (that is, R has finite type).

5.1. Truncated Polynomial Rings in One Variable. Let k be an algebraically closed field of characteristic not two and R the ring $k[x]/x^n k[x]$ with $n \geq 1$. Denote by R_j the ring $k[x]/x^j k[x]$ for $1 \leq j \leq n$. Note we have $R_{j-1} \subseteq R_j$ and $R = R_n$. Let \mathfrak{m} denote the maximal ideal xR of R . Then $\text{End}_R(\mathfrak{m}^i)$ is isomorphic to the local ring R_{n-i} . Let M be the R -module $R \oplus \mathfrak{m} \oplus \cdots \oplus \mathfrak{m}^{n-1}$. We set $E = \text{End}_R(M)$ and seek to compute E_{ab}^* .

For $n = 1$, we have $\text{End}_R(\mathfrak{m}) = k$. Thus $E_{\text{ab}}^* \cong (k^*)^{\oplus 2}$ by ([7], Proposition 9.6).

Suppose now $n \geq 2$. From Proposition 5.1(c), we have a surjection $E_{\text{ab}}^* \rightarrow (k^*)^{\oplus n}$ such that the kernel consists of diagonal matrices $\alpha = (\alpha_{ii})$ with $\alpha_{ii} \in \text{Aut}_R(\mathfrak{m}^{i-1})$. Now every member of $\text{Aut}_R(\mathfrak{m}^{i-1})$ is given by multiplication by an element of R_{n-i+1}^* , thus with Proposition 5.1(b) in hand, we find α_{ii} is given by multiplication by an element of $1 + xR_{n-i+1} \subseteq R^*$. Now every endomorphism on \mathfrak{m}^{n-1} is given by an element of $1 + xR_1 = \{1\}$, so that we can write $\alpha = d_1(\alpha_{11}) \cdots d_{n-1}(\alpha_{n-1, n-1})$. We want to show that α is trivial, so that $E_{\text{ab}}^* \cong (k^*)^{\oplus n}$. It suffices to show each $d_i(\alpha_{ii})$ is in the commutator subgroup of E^* . We do this below.

We show by decreasing induction on i that $d_i(\beta)$ can be written as a product of commutators for β given by multiplication by an element of $1 + xR_{n-i+1}$. For $i = n - 1$, write $\beta = r1_{\mathfrak{m}^{n-2}}$, where $r \in 1 + xR_2$. Notice that $r^{-1} \in 1 + xR_2$ as well, hence multiplication by r^{-1} restricts to the identity on \mathfrak{m}^{n-1} . This gives

$$d_{n-1}(\beta) = d_{n-1}(r1_{\mathfrak{m}^{n-2}}) = d_{n-1}(r1_{\mathfrak{m}^{n-2}})d_n(r^{-1}1_{\mathfrak{m}^{n-1}})$$

Let $\iota_{n-1} : \mathfrak{m}^{n-1} \rightarrow \mathfrak{m}^{n-2}$ be the inclusion. Using this, we have the following factorization of $d_{n-1}(\beta)$

$$e_{n, n-1}((r^{-1} - 1)1_{\mathfrak{m}^{n-2}})e_{n-1, n}(\iota_{n-1})e_{n, n-1}((r - 1)1_{\mathfrak{m}^{n-2}})e_{n-1, n}(-r^{-1}\iota_{n-1})$$

Now we apply ([7], Lemma 9.2) to conclude that $d_{n-1}(\beta)$ is a product of commutators. Suppose now $i < n - 1$ and $\beta \in \text{Aut}_R(\mathfrak{m}^{i-1})$ is given by multiplication on \mathfrak{m}^{i-1} by an element of $1 + xR_{n-i+1}$. We have

$$d_i(\beta) = d_i(\beta)d_{i+1}(\beta^{-1}|_{\mathfrak{m}^i})d_{i+1}(\beta|_{\mathfrak{m}^i})$$

By the induction hypothesis, $d_{i+1}(\beta|_{\mathfrak{m}^i})$ is a product of commutators. Thus it suffices to show $d_i(\beta)d_{i+1}(\beta^{-1}|_{\mathfrak{m}^i})$ is in the commutator subgroup of E^* . Write $\beta = r1_{\mathfrak{m}^{i-1}}$, with $r \in 1 + xR_{n-i+1}$, so that $r^{-1} \in 1 + xR_{n-i+1}$. Let $\iota_i : \mathfrak{m}^i \rightarrow \mathfrak{m}^{i-1}$ be the inclusion. We can factorize $d_i(\beta)d_{i+1}(\beta^{-1})$ as

$$e_{i+1, i}((r^{-1} - 1)1_{\mathfrak{m}^{i-1}})e_{i, i+1}(\iota_i)e_{i+1, i}((r - 1)1_{\mathfrak{m}^{i-1}})e_{i, i+1}(-r^{-1}\iota_i)$$

Where we again apply ([7], Lemma 9.2) to see that $d_i(\beta)d_{i+1}(\beta^{-1})$ is a product of commutators.

5.2. Singularity of Type A_{2n} in Dimension One. Let k be an algebraically closed field of characteristic not equal to 2, 3 or 5 and R the ring $k[[t^2, t^{2n+1}]]$. Set $R = R_0$ and let M be the R -module $R_0 \oplus R_1 \oplus \cdots \oplus R_n$, where $R_i = k[[t^2, t^{2(n-i)+1}]]$ for $i = 0, \dots, n$. We set $E = \text{End}_R(M)$. Before we begin, we make some useful remarks: The R_i are finitely generated R -modules; each R_i is local with maximal ideal $\mathfrak{m}_i = (t^2, t^{2(n-i)+1})R_i$; we have inclusions $R_i \subseteq R_{i+1}$ and $\mathfrak{m}_i \subseteq \mathfrak{m}_{i+1}$; each R_i has k as a residue field; for each $0 \leq i, j \leq n$, any R -linear map of R_j into R_i has the form ufv , where $u \in \text{Aut}_R(R_i)$, $v \in \text{Aut}_R(R_j)$ and f is multiplication by t^a , for $a \in \mathbb{N}$ is such that $t^a R_j \subseteq R_i$ ([21], Proposition 5.11(vi)); and by the proof of ([21], Lemma 9.4), $R_i \cong \text{End}_R(R_i)$ via the map $f \mapsto f1_{R_i}$. We want to show that $E_{\text{ab}}^* \cong (k^*)^{\oplus n} \oplus k[[t]]^*$. We do this below.

This is clear for $n = 0$. For $n = 1$, this is just ([7], Proposition 9.6), since $k[[t]] \cong (t^2, t^3)k[[t^2, t^3]]$ as $k[[t^2, t^3]]$ -modules.

Suppose now $n \geq 2$. We first construct maps from E^* into k^* . To do so, let $\alpha = (\alpha_{ij}) \in E^*$, so that $\alpha_{ij} \in \text{Hom}_R(R_{j-1}, R_{i-1})$. For $1 \leq i < n + 1$, map α to $\alpha_{ii}(1) + \mathfrak{m}_{i-1} \in R_{i-1}/\mathfrak{m}_{i-1} = k$. Since

α is an automorphism of E , we have $\alpha_{ii} \in \text{Aut}_R(R_{i-1})$ by ([7], Proposition 9.4). Now every element of $\text{Aut}_R(R_{i-1})$ is given by multiplication by a unit of R_{i-1} , so that $\alpha_{ii}(1) + \mathfrak{m}_{i-1} \in k^*$. We check this map is a group homomorphism. Let $\beta = (\beta_{ij}) \in E^*$, then

$$(\alpha\beta)_{ii}(1) = \sum_{k=1}^{n+1} (\alpha_{ik}\beta_{ki})(1)$$

Suppose $i \neq k$, so that R_{i-1} and R_{k-1} are nonisomorphic as R -modules. Since both R -modules have local endomorphism rings, ([7], Lemma 9.3) says $(\alpha_{ik}\beta_{ki})(1)$ is a nonunit in $\text{End}_R(R_{i-1}) \cong R_{i-1}$. That is, $(\alpha_{ik}\beta_{ki})(1) \in \mathfrak{m}_{i-1}$. Now $\alpha_{ii}(\beta_{ii}(1)) = \alpha_{ii}(1)\beta_{ii}(1)$, since α_{ii} and β_{ii} are just multiplication on R_{i-1} by an element of R_{i-1}^* . Hence $(\alpha\beta)_{ii}(1) + \mathfrak{m}_i = \alpha_{ii}(1)\beta_{ii}(1) + \mathfrak{m}_i$. Since k^* is abelian, these maps give an induced homomorphism $E_{\text{ab}}^* \rightarrow (k^*)^{\oplus n}$ such that

$$(\alpha_{ij}) \mapsto (\alpha_{11}(1) + \mathfrak{m}_0, \dots, \alpha_{nn}(1) + \mathfrak{m}_{n-1})$$

To construct a map from E^* to $R_n^* = k[[t]]^*$, note that $\alpha_{ij}(1) \in R_{i-1} \subseteq R_n$ for $\alpha_{ij} \in \text{Hom}_R(R_{j-1}, R_{i-1})$. Thus we may consider the composition of the following maps

$$(\alpha_{ij}) \mapsto (\alpha_{ij}(1)) \mapsto \det((\alpha_{ij}(1)))$$

The first map takes $(\alpha_{ij}) \in E$ to the matrix $(\alpha_{ij}(1)) \in \mathbb{M}_{n+1}(R_n)$ and the second takes the matrix $(\alpha_{ij}(1))$ to its determinant in R_n . To see the first map is multiplicative, let $\alpha, \beta \in E$ and note the structure of $\text{Hom}_R(R_{l-1}, R_{m-1})$ and $\text{Hom}_R(R_{m-1}, R_{k-1})$ discussed at the beginning of the proof gives

$$(\alpha\beta)_{kl}(1) = \sum_{m=1}^{n+1} \alpha_{km}\beta_{ml}(1) = \sum_{m=1}^{n+1} \alpha_{km}(1)\beta_{ml}(1) = \left((\alpha_{ij}(1)) \cdot (\beta_{ij}(1)) \right)_{kl}$$

As the determinant is multiplicative, the above composition is multiplicative. Now we show that the above composition maps E^* into R_n^* . We have

$$\det((\alpha_{ij}(1))) = \sum_{\sigma \in S_{n+1}} \left(\text{sgn}(\sigma) \prod_{i=1}^{n+1} \alpha_{i\sigma(i)}(1) \right)$$

Since α_{ii} is an automorphism of R_{i-1} , $\alpha_{ii}(1) \in R_{i-1}^*$. Hence we have $\alpha_{11}(1)\alpha_{22}(1) \cdots \alpha_{n+1,n+1}(1) \in R_n^*$. Thus it suffices to show that when σ is not the identity permutation, $\prod_{i=1}^{n+1} \alpha_{i\sigma(i)}(1) \in \mathfrak{m}_n = tk[[t]] \subseteq k[[t]] = R_n$. If σ is not the identity permutation, choose i minimal such that σ does not fix i , so that $i < \sigma(i)$. Now $\alpha_{i\sigma(i)} : R_{\sigma(i)-1} \rightarrow R_{i-1}$. Then $\alpha_{i\sigma(i)}$ has the form ufv , where u is multiplication on R_{i-1} by an element of R_i^* , v is multiplication on $R_{\sigma(i)-1}$ by an element of $R_{\sigma(i)-1}^*$ and f is multiplication by t^a with $a \geq 0$ such that $t^a R_{\sigma(i)-1} \subseteq R_{i-1}$. Since $i < \sigma(i)$, $a \geq 2$, hence $\prod_{i=1}^n \alpha_{i\sigma(i)}(1) \in \mathfrak{m}_n$. Since R_n^* is abelian, this gives a group homomorphism $E_{\text{ab}}^* \rightarrow R_n^*$ such that $(\alpha_{ij}) \mapsto \det(\alpha_{ij}(1)) \in R_n^*$.

Combining the above, there is a group homomorphism $\Phi : E_{\text{ab}}^* \rightarrow (k^*)^{\oplus n} \oplus R_n^*$ such that $(\alpha_{ij}) \in E_{\text{ab}}^*$ is taken to

$$(\alpha_{11}(1) + \mathfrak{m}_0, \dots, \alpha_{nn}(1) + \mathfrak{m}_{n-1}, \det(\alpha_{ij}(1)))$$

Note Φ is surjective: For $(a_1, \dots, a_n, f) \in (k^*)^{\oplus n} \oplus R_n^*$, (a_1, \dots, a_n, f) is the image under Φ of

$$\text{diag}(a_1 1_{R_0}, a_2 1_{R_1}, \dots, a_n 1_{R_{n-1}}, (a_1 a_2 \cdots a_n)^{-1} f 1_{R_n})$$

To see that Φ is injective, let $\alpha \in E_{\text{ab}}^*$ such that $\Phi(\alpha)$ is trivial. By ([7], Corollary 9.5), we may assume that $\alpha \in E_{\text{ab}}^*$ is diagonal. Write $\alpha = (\alpha_{ii})$, where $\alpha_{ii} = f_{i-1} 1_{R_{i-1}}$ for $i = 1, \dots, n+1$. Since $\Phi(\alpha)$ is trivial, $f_{i-1} \in 1 + \mathfrak{m}_{i-1}$ for $i = 1, \dots, n$ and $f_0 f_1 \cdots f_n = 1$ in R_n^* . Hence for $i = 1, \dots, n$, α is the product of the diagonal automorphisms $\beta_i = d_i(f_{i-1} 1_{R_{i-1}}) d_{n+1}(f_{i-1}^{-1} 1_{R_n})$. For $i < j \leq n$, consider the automorphisms $\gamma_j = d_j(f_{i-1}^{-1} 1_{R_{j-1}}) d_{j+1}(f_{i-1} 1_{R_j})$. If we show each of the γ_j in the commutator subgroup of E^* , in E_{ab}^* we can multiply β_i by the product of the γ_j to obtain that β_i is equivalent to the automorphism $\delta_i = d_i(f_{i-1} 1_{R_{i-1}}) d_{i+1}(f_{i-1}^{-1} 1_{R_i})$ in E_{ab}^* . To see that δ_i is in the

commutator subgroup, note that f_{i-1}^{-1} is in $1 + \mathfrak{m}_{i-1}$, hence multiplication by $f_{i-1}^{-1} - 1$ and $f_{i-1} - 1$ maps R_i into R_{i-1} . Indeed, multiplication by \mathfrak{m}_{i-1} on \mathfrak{m}_i takes \mathfrak{m}_i into \mathfrak{m}_{i-1} . Moreover, any unit in R_i is a power series with nonzero constant term, hence $1 + \mathfrak{m}_{i-1}$ takes R_i into R_{i-1} . This gives the following decomposition of δ_i :

$$[e_{i,i+1}((f_{i-1}^{-1} - 1)1_{R_i})e_{i+1,i}(\iota_i)e_{i,i+1}((f_{i-1} - 1)1_{R_i})e_{i+1,i}(-f_{i-1}^{-1}\iota_i)]^{-1}$$

Where $\iota_i : R_{i-1} \rightarrow R_i$ is the inclusion map. By ([7], Lemma 9.2), each element in the above decomposition is a commutator. Thus it suffices to show that for $i < j \leq n$, γ_j is a commutator. Now multiplication by $f_{i-1}^{-1} - 1$ and $f_{i-1} - 1$ maps R_j into R_{j-1} , hence, we have the following decomposition of γ_j :

$$e_{j,j+1}((f_{i-1}^{-1} - 1)1_{R_j})e_{j+1,j}(\iota_j)e_{j,j+1}((f_{i-1} - 1)1_{R_j})e_{j+1,j}(-f_{i-1}^{-1}\iota_j)$$

Where $\iota_j : R_{j-1} \rightarrow R_j$ is the inclusion map. Again by ([7], Corollary 9.5), we see that $d_j(f_{i-1}^{-1}1_{R_{j-1}})d_{j+1}(f_{i-1}1_{R_j})$ is a product of commutators. Thus Φ is injective, hence an isomorphism.

We collect these results below.

Proposition 5.2. *Let k be an algebraically closed field of characteristic not two. We have the following.*

- (a) *If $R = k[x]/x^n k[x]$ and $M = R \oplus xR \oplus \cdots \oplus x^{n-1}R$, then $\text{Aut}_R(M)_{ab} \cong (k^*)^{\oplus n}$.*
- (b) *If k also has characteristic not equal to 3 or 5, $R = k[[t^2, t^{2n+1}]]$, $n \geq 0$ and $M = R \oplus R_1 \oplus \cdots \oplus R_n$, with $R_i = k[[t^2, t^{2(n-i)+1}]]$, then $\text{Aut}_R(M)_{ab} \cong (k^*)^{\oplus n} \oplus k[[t]]^*$.*

5.3. Invariant Subrings. Let k be a field. Recall from Section 4 that S is the ring $k[[x_1, \dots, x_n]]$, G is a finite subgroup of $GL_n(k)$ that does not contain any nontrivial pseudo-reflections with $|G|$ invertible in k and R is the invariant subring S^G of S (where G acts by a linear change of variables on S). Then if R is an isolated singularity, the R -module S is an $(n-1)$ -cluster tilting object in $\mathbf{mcm} R$. Set $\Lambda = \text{End}_R(S)^{\text{op}}$. While we compute Λ_{ab}^* directly below, we recall that there is an isomorphism $\Lambda^{\text{op}} \cong S \# G$, hence $\Lambda_{ab}^* \cong (S \# G)_{ab}^*$, where $S \# G$ is the skew group ring. However, the task of computing $(S \# G)_{ab}^*$ seems equally as difficult as directly computing Λ_{ab}^* , so this route to Λ_{ab}^* is not so elucidating.

We need the following lemmas for our next example.

Lemma 5.3. *Let A be a local Cohen-Macaulay integral domain of dimension $d > 1$ such that A is an isolated singularity. Then A is normal and $\text{Hom}_A(I, I) \cong A$ for any ideal I of height one.*

Proof. Clearly A satisfies Serre's criterion for normality. For the second part, choose $x \in I$ to be nonzero. Then ([19], Lemma 2.4.3) shows that $\text{Hom}_A(I, I)$ can be identified with the A -submodule $\frac{1}{x}(xI :_A I)$ of the quotient field of A . Now $(Ix :_A I)$ is nonzero and contained in I , so must have height one. If I is principal, it is clear that $(xI :_A I) = xA$. However, as $(xI :_A I)$ has height one and A is an isolated singularity, $A_{\mathfrak{p}}$ is a discrete valuation ring for every associated prime \mathfrak{p} of $(xI :_A I)$, hence $(xIA_{\mathfrak{p}} :_{A_{\mathfrak{p}}} IA_{\mathfrak{p}}) = xA_{\mathfrak{p}}$. Thus $(xI :_A I) = xA$ and $\text{Hom}_A(I, I) \cong A$. \square

Lemma 5.4. ([3], Lemma 5.4)

Let A be a commutative noetherian ring. Then for any ideal I and module M such that $\text{grade}(I, M) \geq 2$, we have $\text{Hom}_A(I, M) \cong \text{Hom}_A(A, M) \cong M$.

5.3.1. Singularity of Type A_1 in Dimension Two. Let R be the A_1 singularity $k[[s^2, st, t^2]]$ in dimension two with $\text{char}(k) \neq 2$. Then R has finite type (so is an isolated singularity), since S is a 1-cluster tilting object for $\mathbf{mcm} R$ and is a ring of invariants (see [11], Example 5.25). Moreover, $\text{End}_R(R \oplus I)^{\text{op}}$ has finite global dimension (see [12], Theorem 2.1) With $S = k[[s, t]]$, the indecomposable maximal Cohen-Macaulay summands of S are R and the ideal $I = (s^2, st)R$ by ([11], Example 5.25). Then I has height one, so that $\text{Hom}_R(I, I) \cong R$ by Lemma 5.3. Moreover,

as I is maximal Cohen-Macaulay, we have $\text{Hom}_R(I, R) \cong R$ by Lemma 5.4. Thus $\text{End}_R(R \oplus I)$ is isomorphic to the subring $\begin{pmatrix} R & R \\ I & R \end{pmatrix}$ of $\mathbb{M}_2(R)$. By ([14], Corollary 2.8), there is an isomorphism

$$\text{Aut}_R(R \oplus I)_{\text{ab}} \cong K_1^C(R) \oplus K_1^C(R/I) = R^* \oplus k[[t^2]]^*$$

Thus if \mathfrak{m} denotes the maximal ideal of R , we have

$$\begin{aligned} R^* \oplus k[[t^2]]^* &\cong k^* \oplus 1 + \mathfrak{m} \oplus k[[t^2]]^* \\ &\cong k^* \oplus k[[t^2]][[s^2, st, t^2]]^* \\ &= k^* \oplus R^* \end{aligned}$$

Thus we have shown.

Proposition 5.5. *Let k be a field. If the characteristic of k is not 2 and $R = k[[s^2, st, t^2]]$, then $\text{Aut}_R(R \oplus (s^2 st)R)_{\text{ab}} \cong k^* \oplus R^*$.*

5.4. Reduced Hypersurface Singularities. Before we provide any other examples, we discuss another route for computing the group $\text{Aut}_R(L)_{\text{ab}}$. We begin with another aside on noncommutative algebra. A ring A with Jacobson radical J is said to be *semiperfect* if A is semilocal and idempotents of A/J lift to idempotents of A . We assume that $\mathbf{mcm} R$ contains an n -cluster tilting object L of the form $L_0 \oplus L_1 \oplus \cdots \oplus L_t$ and L_0, L_1, \dots, L_t are pairwise non-isomorphic and indecomposable. As $\text{End}_R(L_i)$ is local for all i , it is the case that $\Lambda = \text{End}_R(L)^{\text{op}}$ is semiperfect by ([10], Theorem 23.8) (noting that Λ is semiperfect if and only if Λ^{op} is semiperfect). In particular, if R is a k -algebra, the characteristic of k is not two and Λ has finite global dimension, we have the isomorphisms from Lemmas 3.3 and 3.4.

$$\text{Aut}_R(L)_{\text{ab}} = (\Lambda^{\text{op}})_{\text{ab}}^* \cong \Lambda_{\text{ab}}^* \cong K_1(\Lambda) \cong K_1^B(\mathbf{proj} \Lambda) \cong K_1^C(\Lambda)$$

As with previous examples, we have utilized the matrix-like structure of Λ^{op} . That is, for $\alpha \in \Lambda^{\text{op}}$, we write $\alpha = (\alpha_{ij})$, with $\alpha_{ij} : L_{j-1} \rightarrow L_{i-1}$, as i, j run over $1, \dots, t+1$. With the above isomorphisms, we can utilize ([14], Theorem 2.2) to obtain an isomorphism

$$\text{Aut}_R(L)_{\text{ab}} \cong \left(\bigoplus_{i=0}^t \text{Aut}_R(L_i) \right) / HC$$

Where C is the subgroup of $\bigoplus_{i=0}^t \text{Aut}_R(L_i)$ generated by elements of the form

$$(1 + \alpha_{ii}\beta_{ii})(1 + \beta_{ii}\alpha_{ii})^{-1}$$

such that $1_{L_{i-1}} + \alpha_{ii}\beta_{ii} \in \text{Aut}_R(L_{i-1})$ and $1 \leq i \leq t+1$. Suppose $i \neq j$ and we are given $\alpha_{ij} \in \text{Hom}_R(L_{j-1}, L_{i-1})$ and $\alpha_{ji} \in \text{Hom}_R(L_{i-1}, L_{j-1})$. Then $\alpha_{ij}\alpha_{ji}$ and $\alpha_{ji}\alpha_{ij}$ are not automorphisms of L_{i-1} and L_{j-1} , respectively, by ([7], Lemma 9.3). Thus $1 + \alpha_{ij}\alpha_{ji}$ and $1 + \alpha_{ji}\alpha_{ij}$ are automorphisms of L by ([7], Proposition 9.4). Therefore, according to ([14], Theorem 2.2), H is the subgroup generated by

$$(1 + \alpha_{ij}\alpha_{ji})(1 + \alpha_{ji}\alpha_{ij})^{-1}$$

With $1 \leq i \neq j \leq t+1$.

We use the above in our next examples.

5.4.1. Dimension One. Let k be an algebraically closed field of characteristic not 2 and $S = k[[x, y]]$. As in Section 4, R is the ring S/fS and $f \in (x, y)$ is such that in its prime factorization, $f = f_1 \cdots f_n$ we have $f_i \notin (x, y)^2$ for all i . Thus we have the 2-cluster tilting object $L := S_1 \oplus \cdots \oplus S_n$ in $\mathbf{mcm} R$, where $S_i = S/(f_1 \cdots f_i)S$ ([2]). We have the following isomorphisms

$$(\star) \quad \text{Hom}_R(S_j, S_i) \cong \begin{cases} (f_{j+1} \cdots f_i)/(f_1 \cdots f_i) & j < i \\ S_i & i \leq j \end{cases}$$

As $\Lambda := \text{End}_R(L)^{\text{op}}$ has finite global dimension by Theorem 4.1, our preceding work and (\star) gives us the following isomorphism

$$\text{Aut}_R(L)_{\text{ab}} \cong (S_1^* \oplus \cdots \oplus S_n^*)/H$$

Where H is the subgroup of $S_1^* \oplus \cdots \oplus S_n^*$ generated by the elements $h_{ij}(s)$ ($s \in S^*$ and $i < j$) such that

- (i) In the j th position, $h_{ij}(s)$ is the image of an element $s \in 1 + (f_{i+1} \cdots f_j) \subset S$ in the unit group S_j^* .
- (ii) In the i th position, $h_{ij}(s)$ is the image of s^{-1} , with s from (i), in the unit group S_i^* .
- (iii) $h_{ij}(s)$ is trivial elsewhere.

Let H_{ij} be the subgroup of H generated by the $h_{ij}(s)$, with s defined above. We have $H = \bigoplus H_{ij}$, where $1 \leq i < j \leq n$. It is easy to see that H_{ij} is isomorphic to the subgroup $1 + (f_{i+1} \cdots f_j)$ of S_j^* . For $1 \leq i < n$, we call the subgroup $H_{i+1} \oplus \cdots \oplus H_{in}$ of H the i th layer of H . It is easy to see that $S_1^* \oplus \cdots \oplus S_n^*$ modulo the direct sum of the first i layers of H is

$$\bigoplus_{k=1}^{i+1} (S/f_k S)^* \oplus \bigoplus_{l=i+2}^n (S/(f_{i+1} \cdots f_l) S)^*$$

As H is the direct sum of the $n-1$ layers of H , we see that $S_1^* \oplus \cdots \oplus S_n^*$ modulo H is just

$$(S/f_1 S)^* \oplus \cdots \oplus (S/f_n S)^*$$

Thus we have proved the following.

Proposition 5.6. *Let k be an algebraically closed field with characteristic not two and $S = k[[x, y]]$. If $R = S/fS$ and $f \in (x, y)$ is such that in its prime factorization, $f = f_1 \cdots f_n$ we have $f_i \notin (x, y)^2$ for all i , $S_i = S/(f_1 \cdots f_i)S$ and $L = S_1 \oplus \cdots \oplus S_n$, then*

$$\text{Aut}_R(L)_{\text{ab}} \cong (S/f_1 S)^* \oplus \cdots \oplus (S/f_n S)^* = \overline{R}^* \cong G_1(\overline{R})$$

Where $\overline{R} = S/f_1 S \oplus \cdots \oplus S/f_n S$ is the integral closure of R in its total quotient ring.

The isomorphism $G_1(\overline{R}) \cong \overline{R}^*$ requires a little explanation. As $f_i \notin (x, y)^2$, we have that $S/f_i S$ is regular. Thus the additivity of Quillen's K_1 -functor and Quillen's Resolution Theorem give

$$\begin{aligned} G_1(\overline{R}) &= G_1\left(\bigoplus_{i=1}^n S/f_i S\right) \\ &\cong \bigoplus_{i=1}^n G_1(S/f_i S) \\ &\cong \bigoplus_{i=1}^n K_1(S/f_i S) \\ &\cong \bigoplus_{i=1}^n (S/f_i S)^* \\ &= \overline{R}^* \end{aligned}$$

5.4.2. Dimension Three. Keep notation as in the previous subsection with the exception that we now require k be an algebraically closed field of characteristic zero. Set $S' = k[[x, y, u, v]]$ and $R' = S'/(f + uv)S'$ (where f is the product of distinct irreducibles $f_1, \dots, f_n \in (x, y) \setminus (x, y)^2 \subset S = k[[x, y]]$ such that $n > 1$). Consider the 2-cluster tilting object L given by $U_1 \oplus \cdots \oplus U_n$, with $U_i = (u, f_1 \cdots f_i) \subset R'$ (see Section 4). Now R' is Gorenstein of dimension three and an isolated singularity. Since U_i is an ideal of R' of height one, we may apply Lemma 5.3 to see that $\text{Hom}_{R'}(U_i, U_i) \cong R'$ for all i . If $i \neq j$, ([19], Lemma 2.4.3) says we may identify $\text{Hom}_{R'}(U_i, U_j)$ with

the the R' -submodule $\frac{1}{u}(uU_j :_{R'} U_i)$ of the quotient field of R' . Now $(uU_j :_{R'} U_i)$ is nonzero and $(uU_j :_{R'} U_i) \subset U_i$, hence $(uU_j :_{R'} U_i)$ has height one. Let \mathfrak{p} be a minimal prime of $(uU_i :_{R'} U_j)$. As R' is an isolated singularity, $R'_{\mathfrak{p}}$ is a discrete valuation ring. Write $R'_{\mathfrak{p}} = A$ and let μ be a generator for the maximal ideal of A . Suppose u maps to $c\mu^a$, with $a > 0$ and $c \in A^*$. Write $(U_i)_{\mathfrak{p}} = \mu^{n_i}A$ and $(U_j)_{\mathfrak{p}} = \mu^{n_j}A$, with n_j, n_i nonnegative integers. Then

$$(uU_j :_{R'} U_i)_{\mathfrak{p}} = (\mu^{a+n_j} :_A \mu^{n_i})$$

If $i < j$, then $U_j \subset U_i$, hence $n_j \geq n_i$. We have

$$(\mu^{a+n_j} :_A \mu^{n_i}) = \mu^{a+n_j-n_i}A \subset \mu^aA$$

Thus $(uU_j :_{R'} U_i)_{\mathfrak{p}} = (uR')_{\mathfrak{p}}$. In this case, $(uU_i :_{R'} U_j) = uR'$, so that $\text{Hom}_{R'}(U_i, U_j) \cong R'$. Now if $j < i$, then $U_i \subset U_j$ and $n_i \geq n_j$. Notice $u \in U_i$, so that $a \geq n_i$. Now

$$(\mu^{a+n_j} :_A \mu^{n_i}) = \mu^{a+n_j-n_i} = \mu^{a-n_i}(U_j)_{\mathfrak{p}}$$

And $\mu^{a-n_i}A = (\mu^a :_A \mu^{n_i}) = (u :_{R'} U_i)_{\mathfrak{p}}$. We have $(u :_{R'} U_i) = (u :_{R'} f_1 \cdots f_i)$. Thus $(uU_j :_{R'} U_i) = (u :_{R'} f_1 \cdots f_i)U_j$. When $i = n$, $(f_1 \cdots f_n)R' = (uv)R'$, hence $(u :_{R'} f_1 \cdots f_n) = R'$. This gives $U_j = (uU_j :_{R'} U_n)$, hence there is an isomorphism of R' -modules $\text{Hom}_{R'}(U_n, U_j) \cong \frac{1}{u}U_j \cong U_j$.

To analyze the ideal $(u :_{R'} f_1 \cdots f_i)$ for $i < n$, note that $f_{i+1} \cdots f_n \in (u :_{R'} f_1 \cdots f_i)$ and that the ideals $(u, f_{i+1})R', \dots, (u, f_n)R'$ are prime. In particular, the minimal primes of $(u :_{R'} f_1 \cdots f_i)$ are $(u, f_{i+1}R'), \dots, (u, f_n)R'$. Let \mathfrak{q} denote the prime ideal $(u, f_k)R'$, with $i+1 \leq k \leq n$. Then $(f_1 \cdots f_i)R'_{\mathfrak{q}} = R'_{\mathfrak{q}}$, as $f_1, \dots, f_i \notin \mathfrak{q}$ and hence $(u :_{R'} f_1 \cdots f_i)_{\mathfrak{q}} = (uR')_{\mathfrak{q}}$. Thus $(u :_{R'} f_1 \cdots f_i) = uR'$, so that $(u :_{R'} f_1 \cdots f_i)U_j = uU_j$, and hence $\text{Hom}_{R'}(U_i, U_j) \cong U_j$. We collect our results below

$$\text{Hom}_{R'}(U_i, U_j) = \begin{cases} U_j & j < i \\ R' & i \leq j \end{cases}$$

Thus $\text{End}_{R'}(L)$ is the subring of $\mathbb{M}_n(R')$ given by

$$\begin{pmatrix} R' & \cdots & \cdots & \cdots & \cdots & R' \\ U_1 & R' & \cdots & \cdots & \cdots & R' \\ U_1 & U_2 & R' & \cdots & \cdots & R' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_1 & U_2 & U_3 & \cdots & U_{n-1} & R' \end{pmatrix}$$

Noting that $L = U_1 \oplus \cdots \oplus U_n$ and we are using the convention $\alpha \in \text{End}_{R'}(L)$ is given by (α_{ij}) with $\alpha_{ij} : U_j \rightarrow U_i$. By our work at the beginning of this section, we have

$$\text{Aut}_{R'}(L)_{\text{ab}} \cong (R'^*)^{\oplus n} / H$$

We now describe the subgroup H . Here H is the subgroup of $(R'^*)^{\oplus n}$ generated by the elements $h_{ij}(g)$ with $i < j$ and $g \in U_i$ such that that

$$e_k h_{ij}(u) = \begin{cases} 1 + g & k = i \\ (1 + g)^{-1} & k = j \\ 1 & \text{otherwise} \end{cases}$$

with $e_1, \dots, e_n \in R'^{\oplus n}$ the canonical basis. For fixed i and j , let H_{ij} be the subgroup generated by the elements $h_{ij}(g)$. Thus $H = \oplus_{i < j} H_{ij}$ and $H_{ij} \cong 1 + U_i \subset R'^*$. For $i < n$, we call the subgroup $H_{ii+1} \oplus H_{ii+2} \oplus \cdots \oplus H_{in}$ the i th layer of H . As $U_n \subset U_n \subset \cdots \subset U_1$, it is easy to see that $(R'^*)^{\oplus n}$ modulo the direct sum of layers $n-1, n-2, \dots, n-i$ is isomorphic to

$$(R'^*)^{\oplus(n-i)} \oplus (R'/U_{n-i})^{*\oplus i}$$

Now the direct sum of layers $n-1, n-2, \dots, 1$ is just H , so that we see

$$\text{Aut}_{R'}(L)_{\text{ab}} \cong R'^* \oplus (R'/U_1)^{*\oplus(n-1)}$$

Moreover, since $U_1 = (u, f_1)$ and $f_1 \notin (x, y) \setminus (x, y)^2$, we see $R'/U_1 \cong k[[U, V]]$, for variables U, V over k . Thus

$$\mathrm{Aut}_{R'}(L)_{\mathrm{ab}} \cong R'^* \oplus (k[[U, V]]^*)^{\oplus(n-1)}$$

We record these results below.

Proposition 5.7. *Let k be an algebraically closed field of characteristic zero, $S' = k[[x, y, u, v]]$ and $R' = S'/(f + uv)S'$, where $f = f_1 \cdots f_n$ with $f_i \in (x, y) \setminus (x, y)^2$ distinct irreducibles. If $U_i = (u, f_1 \cdots f_i)$ and $L = U_1 \oplus \cdots \oplus U_n$, then*

$$\mathrm{Aut}_{R'}(L)_{\mathrm{ab}} \cong R'^* \oplus (k[[U, V]]^*)^{\oplus(n-1)}$$

Where U, V are variables over k .

6. COMPUTING $G_1(R)$

6.1. The n -Auslander-Reiten Matrix. Before we can completely compute $G_1(R)$, we need to define the free group H occurring in the decomposition of $G_1(R)$. Our assumptions are as usual and we also require that R is a k -algebra and k is algebraically closed of characteristic not two. We use $L = L_0 \oplus L_1 \oplus \cdots \oplus L_t$ to denote an n -cluster tilting object of $\mathbf{mcm} R$ such that $\Lambda = \mathrm{End}_R(L)^{\mathrm{op}}$ has finite global dimension. We assume that L_0, L_1, \dots, L_t are the pairwise non-isomorphic summands of L (each occurs with multiplicity one in the decomposition of L) and that $L_0 = R$. Let $\mathfrak{J} = \{L_0, L_1, \dots, L_t\}$ and $\mathfrak{J}_0 = \mathfrak{J} \setminus \{L_0\}$. We set $\mathcal{C} = \mathbf{add}_R L$. Recall, for $j > 0$, there is an exact sequence, called the n -Auslander-Reiten ending in L_j (see Definition 2.18):

$$0 \longrightarrow C_n^j \longrightarrow \cdots \longrightarrow C_0^j \longrightarrow L_j \longrightarrow 0$$

with $C_0^j, C_1^j, \dots, C_n^j \in \mathcal{C}$. Given $N \in \mathcal{C}$, let $\#(l, N)$ be the number of L_l -summands ($0 \leq l \leq t$) appearing in a decomposition of N into the indecomposables R -modules L_0, L_1, \dots, L_t . Following [13], we define a $(t+1) \times t$ integer matrix T whose lj -th entry is $\#(l, L_j) + \sum_{i=0}^n (-1)^{i+1} \#(l, C_i^j) = \delta_{lj} + \sum_{i=0}^n (-1)^{i+1} \#(l, C_i^j)$ (note that T has a 0th row but no 0th column). As $G_0(k) = \mathbb{Z}$ and $G_0(\Lambda) = \mathbb{Z}^{\oplus t}$ (see Lemma 3.2), Theorem 1.2 gives us a map $\mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$. It is shown in ([13], Section 7.2) that T defines the map $\mathbb{Z}^{\oplus t} \longrightarrow \mathbb{Z}^{\oplus(t+1)}$ afforded to us by Theorem 1.2. We call T the n -Auslander-Reiten matrix or the n -Auslander-Reiten homomorphism. Moreover, this is the same map given in Theorem 1.1 when $\mathbf{mcm} R$ has a 1-cluster tilting object. For our needs, recall Theorem 1.3 says $G_1(R) = H \oplus \mathrm{Aut}_R(L)_{\mathrm{ab}}/\Xi$, so that now $H = \ker(T)$.

We make a useful observation before our computations.

Lemma 6.1. *Let $1 \leq i_1 < \cdots < i_h \leq t$ and $L' = L_{i_1}^{\oplus q} \oplus \cdots \oplus L_{i_h}^{\oplus q}$ with $q > 0$. Then for $a \in R^*$, we have $\det_{\Lambda^{\mathrm{op}}}(\widetilde{a1_{L'}}) = \alpha$, where $\alpha \in (\Lambda^{\mathrm{op}})^*$ is given by $\mathrm{diag}(1_{L_0}, \dots, a^q 1_{L_{i_1}}, \dots, a^q 1_{L_{i_h}}, \dots, 1_{L_t})$*

Proof. From Remark 2.19, we see that $\widetilde{a1_{L'}} : L^{\oplus q} \longrightarrow L^{\oplus q}$ is given by the map $e1_{L^{\oplus q}}$, where $e \in (\Lambda^{\mathrm{op}})^*$ is given by $\mathrm{diag}(1_{L_0}, \dots, a1_{L_{i_1}}, \dots, a1_{L_{i_h}}, \dots, 1_{L_t})$. Now recall the injection $GL_1(\Lambda^{\mathrm{op}}) = (\Lambda^{\mathrm{op}})^* \hookrightarrow GL_q(\Lambda^{\mathrm{op}})$ that takes $\gamma \in (\Lambda^{\mathrm{op}})^*$ to the automorphism $d_1(\gamma) = \mathrm{diag}(\gamma, 1_L, \dots, 1_L) \in GL_q(\Lambda^{\mathrm{op}})$. Now

$$(e1_{L^{\oplus q}})^{-1} \cdot d_1(\alpha) = e^{-1} 1_{L^{\oplus q}} \cdot d_1(\alpha) = \beta_1 \cdots \beta_{q-1}$$

Where $\beta_k = d_1(e) d_{k+1}(e^{-1}) \in GL_q(\Lambda^{\mathrm{op}})$. Thus it suffices to show each β_k is a product of commutators. Now $\beta_1 = d_1(e) d_2(e^{-1})$ and the matrix $\mathrm{diag}(e, e^{-1}) \in GL_2(\Lambda^{\mathrm{op}})$ is a product of commutators in $GL_2(\Lambda^{\mathrm{op}})$ by ([16], Corollary 2.1.3 and Proposition 2.1.4). Since $\beta_1 = \mathrm{diag}(e, e^{-1}, 1, \dots, 1) \in GL_q(\Lambda^{\mathrm{op}})$, we see that β_1 is in the commutator subgroup of $GL(\Lambda^{\mathrm{op}})$. If $q = 2$, we are done. If not, $\beta_1^{-1} \beta_2 = d_2(e) d_3(e^{-1})$ and by the same line of reasoning, we see that $\beta_1^{-1} \beta_2$ is a product of commutators, hence so is β_2 . We can continue inductively, if necessary, by considering the products $\beta_{k-1}^{-1} \beta_k = d_k(e) d_{k+1}(e^{-1})$, where the assumption is that β_{k-1} is a product of commutators in $GL(\Lambda^{\mathrm{op}})$. Thus we see that $e1_{L^{\oplus q}} \equiv d_1(\alpha)$ in $GL(\Lambda^{\mathrm{op}})_{\mathrm{ab}}$.

Since $\det_{\Lambda^{\mathrm{op}}} : GL(\Lambda^{\mathrm{op}})_{\mathrm{ab}} \longrightarrow (\Lambda^{\mathrm{op}})_{\mathrm{ab}}^*$ is the inverse of the isomorphism induced by the map

$$(\Lambda^{\text{op}})^* = GL_1(\Lambda^{\text{op}}) \hookrightarrow GL(\Lambda^{\text{op}}) \twoheadrightarrow GL(\Lambda^{\text{op}})_{\text{ab}}$$

We see that $\det_{\Lambda^{\text{op}}}(e1_{L^{\oplus q}}) = \alpha$. \square

We now make use of Theorem 1.3 to compute $G_1(R)$ for several hypersurface singularities.

6.2. Singularities of Finite Type. Now R has finite type if and only if R has a 1-cluster tilting object M . In this case, $\mathbf{mcm} R = \mathbf{add}_R M$ and $M = M_0 \oplus M_1 \oplus \cdots \oplus M_t$, with $M_0 = R$ and M_1, \dots, M_t the non-free indecomposable maximal Cohen-Macaulay R -modules. For $j > 0$, we call the 1-Auslander-Reiten sequence in M_j the Auslander-Reiten sequences ending in M_j and the 1-Auslander-Reiten matrix is referred to as the Auslander-Reiten matrix. The Auslander-Reiten matrix is a classical invariant and we denote it by Υ .

6.2.1. Truncated Polynomial Rings in One Variable. Let k be an algebraically closed field of characteristic not two and $R = k[x]/x^n k[x]$. Let \mathfrak{m} denote the maximal ideal xR .

For $n = 1$, $R = k$, so $G_1(R) = K_1(R) \cong k^*$.

We now suppose $n \geq 2$. By the proof of ([11], Theorem 3.3), R has finite type and the indecomposable R -modules are given by $R, \mathfrak{m}, \dots, \mathfrak{m}^{n-1}$. Let M be the R -module given by $R \oplus \mathfrak{m} \oplus \cdots \oplus \mathfrak{m}^{n-1}$ and denote its endomorphism ring by E . Using ([21], Lemma 2.9) it is not hard to see the Auslander-Reiten sequences ending in \mathfrak{m}^j are given by

$$0 \longrightarrow \mathfrak{m}^j \longrightarrow \mathfrak{m}^{j-1} \oplus \mathfrak{m}^{j+1} \longrightarrow \mathfrak{m}^j \longrightarrow 0 \quad (1 \leq j \leq n-1)$$

Thus for $1 \leq j \leq n-2$, Υ has its j th column given by $(0, \dots, -1, 2, -1, \dots, 0)^T$, where $-1, 2$ and -1 occur in rows $j-1, j$ and $j+1$, respectively. And the $(n-1)$ st column is given by $(0, \dots, 0, -1, 2)^T$. It is easy to see that Υ is injective.

We compute the subgroup Ξ of E_{ab}^* occurring in Theorem 1.3. By Lemma 6.1, the subgroup Ξ is generated by elements

$$\xi_{a,j} = (\widetilde{a^2 1_{\mathfrak{m}^j}}) \cdot (\widetilde{a^{-1} 1_{\mathfrak{m}^{j-1}} \oplus a^{-1} 1_{\mathfrak{m}^{j+1}}}) \quad (1 \leq j \leq n-2)$$

$$\xi_{a,n-1} = (\widetilde{a^2 1_{\mathfrak{m}^{n-1}}}) \cdot (\widetilde{a^{-1} 1_{\mathfrak{m}^{n-2}}})$$

where a runs over k^* . We have

$$\xi_{a,j} = \text{diag}(1_R, \dots, a^{-1} 1_{\mathfrak{m}^{j-1}}, a^2 1_{\mathfrak{m}^j}, a^{-1} 1_{\mathfrak{m}^{j+1}}, \dots, 1_{\mathfrak{m}^{n-1}})$$

$$\xi_{a,n-1} = \text{diag}(1_R, \dots, a^{-1} 1_{\mathfrak{m}^{n-2}}, a^2 1_{\mathfrak{m}^{n-1}})$$

By Proposition 5.2 (a), there is an isomorphism $E_{\text{ab}}^* \cong (k^*)^{\oplus n}$. We regard Ξ as a subgroup of $(k^*)^{\oplus n}$ and abuse notation to write

$$\xi_{a,j} = (1, \dots, a^{-1}, a^2, a^{-1}, \dots, 1)$$

$$\xi_{a,n-1} = (1, \dots, a^{-1}, a^2)$$

Where a^{-1}, a^2 and a^{-1} occur in $\xi_{a,j}$ at positions $j, j+1$ and $j+2$, respectively. Let $\Psi : (k^*)^{\oplus n} \rightarrow k^*$ be the map such that $\Psi(a_1, \dots, a_n) = a_1^n a_2^{n-1} \cdots a_n$. Then Ψ is a surjective group homomorphism such that $\Xi \subseteq \ker(\Psi)$. Let $(a_1, \dots, a_n) \in \ker(\Psi)$, so that $a_1^n a_2^{n-1} \cdots a_n = 1$. Then $(a_1, \dots, a_n) = \zeta_1 \cdots \zeta_{n-1}$, where

$$\zeta_j = \prod_{r=j}^{n-1} \xi_{a_j^{j-r-1}, r}$$

Thus Ψ induces an isomorphism $\overline{\Psi} : (k^*)^{\oplus n} / \Xi \rightarrow k^*$, hence $G_1(R) \cong k^*$ by Theorem 1.3.

We remark that the isomorphism $G_1(R) \cong k^*$ is immediate from Quillen's Dévissage Theorem ([15], §5 Theorem 4), but we find the process illustrative of our methods as well as allowing us to generalize an example from [7].

6.2.2. Singularity of Type A_{2n} in Dimension One. Let k be an algebraically closed field of characteristic not 2, 3 or 5 and $R = k[[t^2, t^{2n+1}]]$. For $n = 0$, $R = k[[t]]$, a regular local ring, so that $G_1(R) \cong K_1(R) \cong R^* = k[[t]]^*$ by Quillen's Resolution Theorem ([15], §Theorem 3).

We now suppose $n \geq 1$. Now R has finite type and the indecomposable maximal Cohen-Macaulay R -modules are $R_j = k[[t^2, t^{2(n-j)+1}]]$, with $j = 0, \dots, n$ by ([21], Proposition 5.11). Thus M is the R -module $R_0 \oplus R_1 \oplus \dots \oplus R_n$ ($R_0 = R$). Let E be the endomorphism ring of M . By ([21], (5.12)), for $1 \leq j < n$, the Auslander-Reiten sequence ending in R_j is

$$0 \longrightarrow R_j \longrightarrow R_{j-1} \oplus R_{j+1} \longrightarrow R_j \longrightarrow 0$$

And the Auslander-Reiten sequence ending in R_n is

$$0 \longrightarrow R_n \longrightarrow R_{n-1} \oplus R_n \longrightarrow R_n \longrightarrow 0$$

Thus the Auslander-Reiten matrix Υ , for $1 \leq j \leq n-1$, has j th column given by $(0, \dots, -1, 2, -1, \dots, 0)^T$, with $-1, 2$ and -1 occur in rows $j-1, j$ and $j+1$, respectively. The n th column is given by $(0, \dots, 0, -1, 1)^T$. Now Υ is clearly injective, hence $G_1(R) \cong E_{\text{ab}}^*/\Xi$ by Theorem 1.3. We calculate the subgroup Ξ occurring in Theorem 1.3. By Lemma 6.1, the subgroup Ξ is generated by the elements

$$\xi_{a,j} = \widetilde{a^2 1_{R_j}} \cdot a^{-1} 1_{R_{j-1}} \oplus \widetilde{a^{-1} 1_{R_{j+1}}} \quad (1 \leq j < n)$$

$$\xi_{a,n} = \widetilde{a^2 1_{R_n}} \cdot a^{-1} 1_{R_{n-1}} \oplus \widetilde{a^{-1} 1_{R_n}}$$

Where $a \in k^*$. We regard Ξ as a subgroup of $(k^*)^{\oplus(n+1)}$ and compute $(k^*)^{\oplus(n+1)}/\Xi$, viewing the elements of Ξ as a row vectors in $(k^*)^{\oplus(n+1)}$. Hence the elements that generate Ξ are given by

$$\xi_{a,j} = (1, \dots, a^{-1}, a^2, a^{-1}, \dots, 1) \quad (1 \leq j < n)$$

$$\xi_{a,n} = (1, \dots, a^{-1}, a)$$

Where a^{-1}, a^2 and a^{-1} occur in positions $j, j+1$ and $j+2$ for $1 \leq j < n$. Let $\chi : (k^*)^{\oplus(n+1)} \rightarrow k^*$ be given by $\chi(a_1, \dots, a_{n+1}) = a_1 \cdots a_{n+1}$. Then $\ker(\chi)$ consists of (a_1, \dots, a_{n+1}) such that

$$(a_1, \dots, a_{n+1}) = (a_2^{-1}, a_2, 1, \dots, 1)(a_3^{-1}, 1, a_3, 1, \dots, 1) \cdots (a_{n+1}^{-1}, 1, 1, \dots, a_{n+1})$$

We show $\Xi = \ker(\chi)$. Obviously, $\Xi \subseteq \ker(\chi)$. For the converse, it suffices to show the elements $\zeta_{a,j} = (a^{-1}, 1, \dots, a, \dots, 1)$, where a is in the j th position and $2 \leq j \leq n+1$, are in Ξ . Indeed, note that $\zeta_{2,a} = \xi_{a,1} \xi_{a,2} \cdots \xi_{a,n}$ and for $j > 2$, we have $\zeta_{a,j} = \zeta_{a,j-1} \xi_{a,j-1} \xi_{a,j} \cdots \xi_{a,n}$. Thus $\ker(\chi) = \Xi$ as needed.

Combining the above and using (b) of Proposition 5.2, we have

$$G_1(R) \cong (k^*)^{\oplus(n+1)}/\Xi \oplus (1 + tk[[t]]) \cong k^* \oplus (1 + tk[[t]]) \cong k[[t]]^*$$

6.2.3. Singularity of Type A_1 in Dimension Two. Let k be an algebraically closed field of characteristic not two and R the ring $k[[s^2, st, t^2]]$. By ([11], Example 5.25 and 13.21) R has finite type and the indecomposable maximal Cohen-Macaulay R -modules are R and $I = (s^2, st)$. Moreover, the Auslander-Reiten sequence ending in I is given by

$$0 \longrightarrow I \longrightarrow R^2 \longrightarrow I \longrightarrow 0$$

Set $M = R \oplus I$ and let E be its endomorphism ring.

An easy calculation shows that the Auslander-Reiten homomorphism $\Upsilon : \mathbb{Z} \rightarrow \mathbb{Z}^{\oplus 2}$ is injective. Now Ξ is the subgroup of E_{ab}^* generated by the elements

$$\widetilde{a1_I} \cdot \det_E(\widetilde{a1_{R^2}})^{-1} \cdot \widetilde{a1_I} = \widetilde{a^2 1_I} \cdot \det_E(\widetilde{a1_{R^2}})^{-1} \quad (a \in k^*)$$

The automorphism of M , $\widetilde{a^2 1_I}$, is given by $\text{diag}(1_R, a^2 1_I)$. Using Lemma 6.1, $\det_E(\widetilde{a1_{R^2}})$ is the image of the automorphism $\text{diag}(a^2 1_R, 1_I)$ in E_{ab}^* . Thus Ξ is the subgroup of E_{ab}^* generated by the elements

$$\widetilde{a^2 1_I} \cdot \det_E(\widetilde{a1_{R^2}})^{-1} = \text{diag}(1_R, a^2 1_I) \cdot \text{diag}(a^{-2} 1_R, 1_I) = \text{diag}(a^{-2} 1_R, a^2 1_I)$$

As groups, $\Xi \cong k^{*2} = \{a^2 : a \in k^*\}$. Since k is algebraically closed, $k^{*2} = k^*$. Using Proposition 5.5, we have $E_{\text{ab}}^* \cong k^* \oplus R^*$, hence $E_{\text{ab}}^*/\Xi \cong R^*$. Thus $G_1(R) \cong R^*$ by Theorem 1.3, since Υ is injective.

6.3. Reduced Hypersurface Singularities.

6.3.1. Dimension One. Suppose k is an algebraically closed field of characteristic not two and $S = k[[x, y]]$. As previously noted, if $S = k[[x, y]]$, $f_1, \dots, f_n \in (x, y) \setminus (x, y)^2$, with f_i irreducible, and $R = S/(f_1 \cdots f_n)S$ is an isolated singularity (ie. $f_i S \neq f_j S$), then $L = S_1 \oplus \cdots \oplus S_n$, with $S_i = S/(f_1 \cdots f_i)$, is a 2-cluster tilting object in $\mathbf{mcm} R$. In order to compute $G_1(R)$, we need to understand the structure of the 2-Auslander-Reiten sequences in $\mathcal{C} = \mathbf{add}_R L$. To do this, we require the additional assumption that $(f_i, f_{i+1}) = (x, y)$. In this case, by ([8], Proof of Theorem 4.11) the structure of the 2-Auslander-Reiten sequences are

$$0 \longrightarrow S_j \longrightarrow S_{j+1} \oplus S_{j-1} \longrightarrow S_{j+1} \oplus S_{j-1} \longrightarrow S_j \longrightarrow 0 \quad (1 \leq j < n)$$

From this and Lemma 6.1 it is clear that the subgroup Ξ of $\text{Aut}_R(L)_{\text{ab}}$ is trivial. Thus $G_1(R) \cong \ker(T) \oplus \text{Aut}_R(L)_{\text{ab}} \cong \ker(T) \oplus \overline{R}^*$, where T is the 2-Auslander-Reiten matrix. Now $T : \mathbb{Z}^{\oplus(n-1)} \longrightarrow \mathbb{Z}^{\oplus n}$ and it is easy to see that T is the zero matrix. This yields

$$G_1(R) \cong \mathbb{Z}^{\oplus(n-1)} \oplus \overline{R}^*$$

6.3.2. Dimension Three. Let k be an algebraically closed field of characteristic zero, $S' = k[[x, y, u, v]]$ and $R' = S'/(f + uv)S'$, where $f = f_1 \cdots f_n$ with $f_i \in (x, y) \setminus (x, y)^2$ distinct irreducibles. If $U_i = (u, f_1 \cdots f_i)$, then $L = U_1 \oplus \cdots \oplus U_n$ is a 2-cluster tilting object. Then by ([13], Proposition 7.28), the 2-Auslander-Reiten matrix T is zero. By Proposition 5.7, $\text{Aut}_{R'}(L)_{\text{ab}} \cong R'^* \oplus (k[[U, V]])^{*\oplus(n-1)}$ (U and V variables over k), thus Theorem 1.3 yields

$$G_1(R') \cong \mathbb{Z}^{\oplus(n-1)} \oplus \left(R'^* \oplus (k[[U, V]])^{*\oplus(n-1)} \right) / \Xi$$

Where Ξ is the subgroup of $R'^* \oplus (k[[U, V]])^{*\oplus(n-1)}$ occurring in Theorem 1.3.

We collect the previously unknown results below.

Theorem 6.2. *Let k be an algebraically closed field of characteristic not two. Then*

(a) *if the characteristic of k is also not 3 or 5 and R is the finite-type singularity A_{2n} in dimension one, $G_1(R) \cong \overline{R}^*$.*

(b) *if the characteristic of k is not 2 and R is the finite-type singularity A_1 in dimension two, then $G_1(R) \cong R^* = \overline{R}^*$.*

(c) *if $S = k[[x, y]]$ and $f_1, \dots, f_n \in (x, y)$ are irreducibles whose product is f are such that*

(i) *$R := S/fS$ is an isolated singularity (ie. $(f_i) \neq (f_j)$)*

(ii) *$f_i \notin (x, y)^2$ for all i .*

(iii) *$(f_i, f_{i+1}) = (x, y)$.*

Then $G_1(R) \cong \mathbb{Z}^{\oplus(n-1)} \oplus \overline{R}^$.*

(d) *If k has characteristic zero, $S' = k[[x, y, u, v]]$ and $R' = S'/(f + uv)S'$, where $f = f_1 \cdots f_n \in k[[x, y]]$ is as in (d) and satisfies (i)-(ii), then $G_1(R') \cong \mathbb{Z}^{\oplus(n-1)} \oplus (R'^* \oplus (k[[U, V]])^{*\oplus(n-1)}) / \Xi$, with Ξ the subgroup from Theorem 1.3.*

7. DISCUSSION

It is of interest to note that in (a)-(c) of Theorem 1.3, $G_1(R)$ contains \overline{R}^* (\overline{R} its the integral closure of R in its total quotient ring) as a direct summand. Our methods were ad hoc and tailored specifically to each singularity via the calculation of the group $\Lambda_{\text{ab}}^* = \text{Aut}_R(L)_{\text{ab}}$, so a deeper look into the relationship between $\Lambda = \text{End}_R(L)^{\text{op}}$ and \overline{R} could shed some light on the structure of $G_1(R)$ for hypersurface singularities.

In fact, the key to the relationship seems to be understanding the relationship between the derived categories of $\mathbf{mod} \Lambda$ and $\mathbf{mod} \overline{R}$. Indeed, in [5], it is shown that if A and B are noetherian rings whose derived categories are equivalent as triangulated categories, then there is an isomorphism and $G_i(A) \cong G_i(B)$ for $i \geq 0$. Of course, one should not expect an equivalence of the derived categories of $\mathbf{mod} \Lambda$ and $\mathbf{mod} \overline{R}$ since our examples indicate $G_1(\mathbf{mod} \Lambda) \cong \Lambda_{\text{ab}}^*$ only contains \overline{R}^* as a direct summand. Moreover, it may also be too much to ask that $G_1(\overline{R})$ is a direct summand of $G_1(\Lambda)$, as $G_1(\overline{R})$ is not always isomorphic to \overline{R}^* . However, if R is a reduced one-dimensional local noetherian ring, then $\overline{R} = \overline{R/\mathfrak{p}_1} \times \cdots \times \overline{R/\mathfrak{p}_s}$, where the \mathfrak{p}_j are the minimal primes of R and each ring $\overline{R/\mathfrak{p}_j}$ is a semilocal principal ideal domain. In this situation

$$G_i(\overline{R}) \cong G_i(\overline{R/\mathfrak{p}_1}) \times \cdots \times G_i(\overline{R/\mathfrak{p}_s})$$

Now $\overline{R/\mathfrak{p}_j}$ is semilocal and has finite global dimension, hence if R is an algebra over a field k with $\text{char}(k) \neq 2$, then $G_1(\overline{R/\mathfrak{p}_j}) \cong (\overline{R/\mathfrak{p}_j})^*$ by Lemma 3.4. Thus $G_1(\overline{R}) \cong \overline{R}^*$ in this case. We conjecture that if R satisfies the hypotheses of Theorem 1.3, then $G_1(R)$ contains \overline{R}^* as a direct summand.

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